Ion Optical Study of Particle Spectrometer System in Higher Order Approximation

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[1] Introduction

Particle analyzers, for example, mass spectrometers, β-ray spectrometers, on-line mass separators, etc., are very useful instruments for the investigation in nuclear science. With the recent progress of nuclear physics in both theoretical and experimental aspects, more accurate and new experiments which use particle analyzers are going to be required.

For such a purpose ion optical calculation plays an important role. Many problems in ion optics are essentially reduced to those of solving the equations of motion of a charged particle in the electromagnetic field in a paraxial approximation and their results are usually expressed as the form of transfer matrix similar to light optics.

The electric fields or the magnetic fields which are used as particle analyzers have normally the property of axial symmetry with respect to some center axis and also of mirror symmetry with respect to a plane perpendicular to the center axis. The plane of symmetry is called the "median plane". Along a circle in the median plane which is concentric to the axis of rotation the field strength is expected to be constant, and therefore some special particles can move along a circle having a radius of curvature ρ_0 because of the balance of centrifugal force and electromagnetic force. Such a trajectory is called the "main path".

If we define the coordinates (x,y) and the angles of inclination α,β of an arbitrary ray with respect to the main path (see fig.1,p10) and assume the quantities $(\frac{x}{\rho_0}, \frac{y}{\rho_0}, \alpha, \beta)$ are small compared to unity, the trajectory of any arbitrary ray can be expressed as a power series of these quantities at the initial position, the coefficients of which being functions of the position along the main path. Since ions in a group are expected to have slightly different energy or momentum, the fifth quantity δ (energy or momentum deviation with respect to the main path particle) is necessary in addition to the four quantities described above.

In the early stage of ion optics, the instruments were designed using the calculation in a first order approximation of five quantities (x, y, α, β, δ), and the relevant ion optical calculations were published by many authors[1).]

In the next step the second order calculation were developed[2)] and on the basis of these results some examples of spectrometer system which aim the second order focusing have been proposed. Many big machines have been constructed but the results are not so good as expected in many cases.

In the high resolution mass spectrometer system, the energy focusing is also necessary in addition to the directional focusing, and the perfect second order focusing of this system is becoming important. The meaning of "perfect" is that all the second order coefficients concerning with angles and energy should be diminished.

When determining the atomic masses very precisely, peak matching method is widely adopted for the measurement of mass difference of a mass doublet.[3)]

In this method the mass ratio of two component ions of doublet is converted to the voltage ratio of the electrodes assuming that the trajectories of two kinds of ions coincide exactly with each other. Even if two ion beams are found at the same position at the detector, the trajectories in the intermediate region may be different to each other and some systematic error may be introduced. This is true if the double(angle and energy) focusing is not complete, but the trouble will be eliminated if the focusing is good. With the increase of the accuracy of measurement the influence of the second order aberration can not be neglected.

In order to achieve the second order focusing it is necessary to estimate the third order influence, because the remaining big effect comes from third order terms after the second order terms are reduced.

The third order calculation makes it possible to estimate the second order influence of the ion beam moving along the noncircular main path. In the conventional calculations, it is expected that the central part of the beam moves along the circular main path, i.e., the geometrical center of the field. From the practical view point, however, the beam center may take different path each time experimental condition is changed and in many cases the beam center may not coincide with the circular geometrical center.

In order to get more complete and useful knowledge of final image position and image aberrations, ion trajectory calculation under such circumstances that fit with the real experimental condition must be accomplished.

By using the results of third order calculation along normal circular main path, above estimation relative to noncircular main path become possible to the second order precision. Making good use of this nature inversely, an excellent method can be derived theoretically how to reduce image aberrations of the already fixed spectrometer system by changing the experimental conditions , for example, the relative strength of the two fields.

With the advance of accelerator, it becomes possible to produce many new nuclides that are far off the stability line. The number of such nuclides is expected more than three thousands. In order to determine the atomic masses of such nuclides, on-line mass spectrometer is considered to be a suitable instrument. The instrument for such a purpose must possess high resolving power, high luminosity and less aberrations. In this aspect the third order calculation is very important and effective arms.

From the reasons described above, the author executes the third order calculations of ion trajectory in both toroidal electric field and inhomogeneous magnetic field. The results are arranged in the form of transfer matrix and the electric computer program for the calculation of third order ion trajectory has been accomplished. The program works quite satisfactory by only introducing the necessary parameters. The product of third order fringing field matrices for the electric and magnetic field[6)] describes the radial motion of charged particle through the real electric and magnetic sector field order.

The way how to derive the new transfer matrix relative to noncircular main path is also established including the computer program. Applying these calculations to r^{-1} high resolution spectrometer in our laboratory we have obtained the reasonable results about the focusing property.

[2] Fermat's Principle and Path Equations in the Electromagnetic Field

2.1. Variational Principles and Euler-Lagrange Equation

The use of superlative enables one to express in concise form a general principle covering a wide variety of phenomena. The statement that a physical system so acts that some function of its behavior is least (or greatest) is often both the starting point for theoretical investigation and the ultimate distillation of all the relationships between facts in a large segment of physics. The mathematical formulation of the superlative is usually that the integral of some function, typical of the system, has a smaller (or else larger) value for the actual performance of the system than it would have for any other imagined performance subject to the same very general requirements which serve to particularize the system under study. We can call the integrand J ; it is a function of a number of independent variables of the system (generalized coordinates) and of the derivatives of these variables with respect to the parameters of integration (generalized velocities). If the variables are ψ_1, ψ_2, ·····, ψ_n, the derivaties $\dot{\psi}_1'$, $\dot{\psi}_2'$, ·····, $\dot{\psi}_n'$ and parameter χ, then the integral which is to be minimized is

$$\mathcal{J} = \int J[\ \psi_1(\chi),\ \psi_2(\chi),\ \cdots\ \psi_n(\chi),\ \dot{\psi}_1(\chi),\ \cdots \dot{\psi}_n(\chi),\ \chi\ ]d\chi \qquad (2\text{-}1)$$

From the minimization of this function we can obtain the partial differential equations governing ψ's as functions of the χ's. These differential equations are derived according to the viriational method and are expressed as :

$$\frac{\partial J}{\partial \psi_i} - \frac{d}{d\chi}\left(\frac{\partial J}{\partial \dot{\psi}_i}\right) = 0\ , \qquad i=1,2,\ \cdots\ n\ . \qquad (2\text{-}2)$$

These equations, which serve to determine the optimum functional form of the ψ's, are called the Euler-Lagrange differential equations.

2.2. Hamilton's Principle and Lagrange's Equation

We define a new function, the Lagrangian L, as

$$L = T - V \qquad (2\text{-}3)$$

identifying T with the system kinetic energy and V with the generalized potential. When the system is conservative (in the larger sense, admitting generalized potential), the variational principle determines the equations of motion, called "Hamilton's Principle". We can summarize Hamilton's principle by saying the motion is such that the variation of line integral of L the "Lagrangian" for fixed t_1 and t_2 is zero, i.e.,

$$\delta\mathcal{L} = \delta\int_{t_1}^{t_2} L(q_1, \cdots q_n, \dot{q}_1, \cdots \dot{q}_n, t)\,dt = 0 \qquad (2\text{-}4)$$

Hamilton's principle has just the form stipulated in eq. (2-2) with the transformations

$$\chi \to t$$
$$\psi_i \to \dot{q}_i$$
$$\dot{\psi}_i \to q_i$$
$$J(\psi_i, \dot{\psi}_i, x) \to L(q_i, \dot{q}_i, t)$$

The Euler-Lagrange equations then become as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i=1,2, \cdots\cdots n \qquad (2\text{-}5)$$

This represents the well known "Lagrange's equations" which will determine the equations of motion.

2.3. Fermat's Principle and Path Equation

Another variational principle associated with the Hamiltonian formulation is known as the Fermat's principle. Which is defined as

$$\delta\int_{s_1}^{s_2} n\,ds = 0 \qquad (2\text{-}6)$$

where n and ds represents ion optical refractive index and the arc length of the particle trajectory ,respectively. Then the term "n ds" equals optical path. Fermat's principle states that of all paths possible between two points, consistent with conservation of energy, the system moves along that particular path for which the "optical path" of transit is the least (more strictly, an extremum).

We will derive this principle from Hamilton's principle. At first we introduce a new function H from Lagrangian by the Legendre transformation :

$$H(p_i, q_i, t) = \sum_i p_i\dot{q}_i - L(q_i, \dot{q}_i, t) \quad , \qquad (2\text{-}7)$$

where $$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad . \qquad (2\text{-}8)$$

H is known as the "Hamiltonian" whose physical significance is that if L (and in consequence of eq.(2-7), also H) is not an explicit function of t then H is a constant of the motion. If potential energy is independent of velocity, H is just equal to the total energy of the system.

Introduction of velocity - dependent potential into the Hamiltonian formulation poses no formal difficulty, but it is no longer clear whether H represents the total energy. However, for the particular case of electromagnetic forces, the direct application of the definition of H, eq.(2-7) proves that the Hamiltonian can still be written as a sum of kinetic and potential energies.

The Lagrangian (nonrelativistic) for a single particle is

$$L = \frac{1}{2} mv^2 - e\Phi + e\, A \cdot V \qquad (2\text{-}9)$$

with canonical momenta

$$p_i = mv_i + eA_i \qquad (2\text{-}10)$$

From the definition, H is given by

$$\begin{aligned} H &= \sum p_i v_i - L \\ &= mv^2 + e\, A \cdot V - L \end{aligned} \qquad (2\text{-}11)$$

or finally

$$H = \frac{1}{2} mv^2 + e\Phi = T + e\Phi \quad , \qquad (2\text{-}12)$$

this represents the total energy of the particle.

2.4. Derivation of Fermat's Principle from Hamilton's Principle

Hamilton's principle equals :

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \quad .$$

Expressing L in terms of the Hamiltonian by eq.(2-7)

$$\delta \int_{t_1}^{t_2} \sum p_i \dot{q}_i dt - \delta \int_{t_1}^{t_2} H\, dt = 0 \quad . \qquad (2\text{-}13)$$

By eq.(2-12) H is conservative

$$\delta\int_{t_1}^{t_2} H\, dt = 0 \quad .$$

Then
$$\delta\int_{t_1}^{t_2} \sum p_i \dot{q}_i dt = 0$$

using the relation $\dot{q}_i dt = \frac{dq_i}{dt} dt = dq_i$,

$$\delta\int \sum p_i dq_i = 0 \quad . \tag{2-14}$$

Let U_1, U_2, U_3 be unit vectors in the direction dS_1, dS_2, dS_3, respectively then we have

$$\sum\frac{\partial L}{\partial \dot{q}_i} dq_i = \sum \frac{1}{Q_i}\frac{\partial L}{\partial \dot{q}_i} Q_i dq_i = \sum \frac{1}{Q_i}\frac{\partial L}{\partial \dot{q}_i} ds_i \quad , \tag{2-15}$$

where Q_i is a "scale factor" for the coordinate q_i .
On the other hand

$$v_i = \frac{ds_i}{dt} = \frac{Q_i dq_i}{dt} = Q_i \dot{q}_i \quad ,$$

$$V = v_1 U_1 + v_2 U_2 + v_3 U_3 \quad ,$$

Then
$$v^2 = v_1{}^2 + v_2{}^2 + v_3{}^2 = Q_1{}^2\dot{q}_1{}^2 + Q_2{}^2\dot{q}_2{}^2 + Q_3{}^2\dot{q}_3{}^2 \quad .$$

Introducing these results into eq.(2-9)

$$L = \frac{m}{2}(Q_1{}^2\dot{q}_1{}^2 + Q_2{}^2\dot{q}_2{}^2 + Q_3{}^2\dot{q}_3{}^2) - e\Phi(q_1, q_2, q_3)$$
$$+ e(Q_1\dot{q}_1A_1 + Q_2\dot{q}_2A_2 + Q_3\dot{q}_3A_3) \tag{2-16}$$

and
$$\frac{\partial L}{\partial \dot{q}_i} = mQ_i{}^2\dot{q}_i + eQ_iA_i \quad , \tag{2-17}$$

$$\frac{1}{Q_i}\frac{\partial L}{\partial \dot{q}_i} = mv_i + eA_i \quad . \tag{2-18}$$

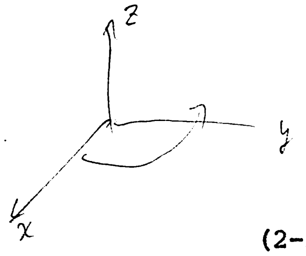
We define the unit vector in the tangential direction of path t as follows :

$$t = \frac{ds_1}{ds}U_1 + \frac{ds_2}{ds}U_2 + \frac{ds_3}{ds}U_3 \tag{2-19}$$

Introducing eq.(2-18) into eq.(2-15) .

$$\sum p_i dq_i = \sum (mv_i + eA_i) ds_i$$

$$= \sum (mv_i \frac{ds_i}{ds} + eA_i \frac{ds_i}{ds}) ds$$



$$= (m\,\mathbb{V}\cdot + e\,\mathbb{A}\cdot)\,ds \qquad (2\text{-}20)$$

From eq.(2-13) and eq.(2-20)

$$\delta \int \mathbb{p}\cdot d\mathbb{r} = 0$$

$$\delta\int_{t_1}^{t_2} L dt = \delta\int_{s_0}^{s_1} (mv + e\,\mathbb{A}\cdot) ds = 0 \qquad (2\text{-}21)$$

$$\delta \int (p\cdot dl + e\mathbb{A}\cdot d\mathbb{r}) = 0$$

This expression is nothing but Fermat's principle. ($\mathbb{p}\cdot d\mathbb{r} = p\cdot dl$)
We define "an ion optical refractive index" n as

$$n = mv + e\,\mathbb{A}\cdot \qquad (2\text{-}22)$$

$$\delta \int \left\{ \sqrt{\left(\frac{\mathcal{E} - e\phi}{c}\right)^2 - m^2c^2}\; dl + e\mathbb{A}\cdot d\mathbb{r} \right\} = 0$$

n means "an ion optical refractive index".

In order to execute the calculation we will use here cylindrical coordinate. In this case

$$mv\, ds = mv\sqrt{r^2 + (\frac{dr}{d\omega})^2 + (\frac{dz}{d\omega})^2}\, d\omega \quad , \qquad (2\text{-}23)$$

and if the fields do not depend on ω, we obtain

$$e\,\mathbb{A}\cdot\mathbb{T}\, ds = e\, rA_\omega\, d\omega \quad .$$

Then

$$n\, ds = \left[mv\sqrt{r^2 + (\frac{dr}{d\omega})^2 + (\frac{dz}{d\omega})^2} + erA_\omega \right] d\omega \quad . \qquad (2\text{-}24)$$

It is necessary to know the relation between the magnitude of kinetic momentum (mv) and the accelerating energy U. According to eq.(2-12)

$$\frac{1}{2} mv^2 + e\Phi = U \quad , \qquad (2\text{-}25)$$

then

$$mv = \sqrt{2m(U - e\Phi)} \quad .$$

As in fig. 2,p21 we choose variables (x, ω, y) instead of (r, ω, z) in cylindrical coordinates as

$$x = r - \rho_0$$
$$\omega = \omega \qquad (2\text{-}26)$$
$$y = z$$

Introducing these notations into eq.(2-25)

$$nds = \rho_0\sqrt{2mU}\left\{\sqrt{\frac{(U-e\Phi)}{U}\left[\left(1+\frac{x}{\rho_0}\right)^2+\left(\frac{x'}{\rho_0}\right)^2+\left(\frac{y'}{\rho_0}\right)^2\right]} + \frac{e}{\sqrt{2mU}}\left(1+\frac{x}{\rho_0}\right)A_\omega\right\}d\omega \qquad (2\text{-}27)$$

Fermat's principle can be rewritten

$$\delta\int F d\omega = 0 \qquad (2\text{-}28)$$

$$F = \sqrt{\frac{(U-e\Phi)}{U}\left[\left(1+\frac{x}{\rho_0}\right)^2+\left(\frac{x'}{\rho_0}\right)^2+\left(\frac{y'}{\rho_0}\right)^2\right]} + \frac{e}{\sqrt{2mU}}\left(1+\frac{x}{\rho_0}\right)A_\omega \qquad (2\text{-}29)$$

where x' and y'stands for $dx/d\omega$ and $dy/d\omega$, respectively. As we discussed in section (2.2.), eq.(2-28) may be written also in the form of Euler-Lagrange Equation,

$$\frac{d}{d\omega}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0 \qquad (2\text{-}30a)$$

$$\frac{d}{d\omega}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \quad . \qquad (2\text{-}30b)$$

We have accomplished our original aim, to derive the path equation in the electromagnetic field.

We have found that there are two kinds of equations, eqs.(2-5) and eqs.(2-30) which determine the same ion trajectory in the electromagnetic field. In the former case, the independent variable is time, while, in the latter case it is angle. With the transformations

$$\frac{d}{dt} = \frac{d}{d\omega}\cdot\frac{d\omega}{dt} \qquad (2\text{-}31)$$

$$\frac{d^2}{dt^2} = \frac{d^2}{d\omega^2}\left(\frac{d\omega}{dt}\right)^2 + \frac{d}{d\omega}\,\frac{d^2\omega}{dt^2} \qquad (2\text{-}32)$$

both equations should coincide. We will derive the solution in the electric field in accordance with Euler-Lagranges equation and that of the magnetic field by Lagrange's equation.

[3] Transfer Matrix and Ion Optical Position Vector

In this chapter we will state under what initial condition we shall solve the path equations given in eqs.(2-30) and how we shall arrange the solution for the purpose of practical use. It is well known that in the paraxial approximation of optics a frequent use is made of matrix operations. Because of the great analogy between light optics and ion optics this type of operations is also applicable in ion optics.

Consider two planes perpendicular to the main ion trajectory. (see fig.1) The first plane, having coordinates (x_2, y_2) is acting as an object plane, the second plane has coordinates (x_3, y_3). The trajectory of an ion passing through the x_2, y_2 plane is projected on to the symmetry plane and also on to the plane through the main orbit, perpendicular to the symmetry plane. These projections make angles α_2 and β_2 with the main orbit. Then the initial conditions are defined by four quantities (x_2, y_2, α_2, β_2). The second plane, having coordinates (x_3, y_3, α_3, β_3), is acting as an image plane.

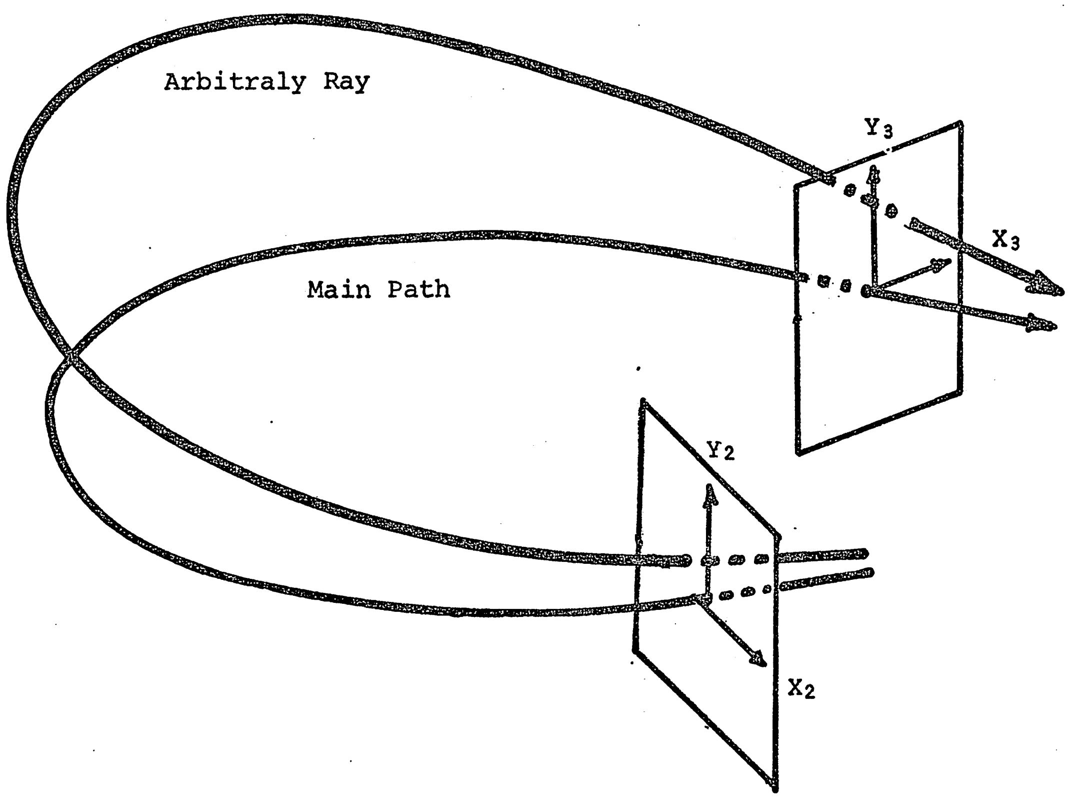


Fig.1. Coordinate system.

We confine ourselves to paraxial trajectories. The trajectories of an ion with a certain momentum and energy will be defined by $(x_2, y_2, \alpha_2, \beta_2)$. Thus x_3, y_3 are also determined by these four quantities. Hence in a first order approximation one may write :

$$x_3 = a_0 + a_1x_2 + a_2y_2 + a_3\alpha_2 + a_4\beta_2 \quad (3\text{-}1a)$$

$$y_3 = b_0 + b_1x_2 + b_2y_2 + b_3\alpha_2 + b_4\beta_2 \quad . \quad (3\text{-}1b)$$

Since the trajectory should be symmetric with respect to the median plane, we can eliminate some coefficients. Then eqs.(3-1) are written as :

$$x_3 = a_0 + a_1x_2 + a_3\alpha_2 \quad (3\text{-}2a)$$

$$y_3 = b_2y_2 + b_4\beta_2 \quad (3\text{-}2b)$$

Considering the later results of eqs.(5-17) which will be derived by solving the differential equation, we can assure the reasonability of eqs.(3-2) and find that the coefficient a_0 is related to energy or momentum deviation δ. We can get the similar expression for the inclination angle α_3 and β_3.

From eq.(3-2) it can be seen the relationships are given by linear combinations. If a small variation in these quantities is admitted the position and the direction of the orbit in image space will be effected. In an electric field the ion orbit will be influenced by the energy of the particle, in a magnetic field by its momentum, in a mass spectrometer by its mass. We use the capital δ as this deviation.

Then the path of the ion passing through the (x_3, y_3) plane is finally given by the relationships:

$$x_3 = R_{11}x_2 + R_{12}\alpha_2 + R_{13}\delta \quad (3\text{-}3a)$$

$$\alpha_3 = R_{21}x_2 + R_{22}\alpha_2 + R_{23}\delta \quad (3\text{-}4a)$$

$$y_3 = Z_{11}y_2 + Z_{12}\beta_2 \quad (3\text{-}3b)$$

$$\beta_3 = Z_{21}y_2 + Z_{22}\beta_2 \quad (3\text{-}4b)$$

where R_{ij} and Z_{ij} is a quantity determined by the properties of the field, for example, field constant and angle of deflection. If the properties which an ion has at the entry into the field are expressed by vectors (x_2, α_2, δ), (y_2, β_2) and those at the exit are expressed by vectors (x_3, α_3, δ), (y_3, β_3). (δ is regarded as unvarying), then a single sector type field may be regarded as producing a transformation from one vector to the other.

We define this vector as an "Ion Optical Position Vector". Accordingly this field can be expressed by the 3-row, 3-column and 2-row, 2-column matrices.

The two vectors are connected by the product relationship

$$\begin{pmatrix} x_3 \\ \alpha_3 \\ \delta \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ \alpha_2 \\ \delta \end{pmatrix} \tag{3-5}$$

$$\begin{pmatrix} y_3 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} y_2 \\ \beta_2 \end{pmatrix} \tag{3-6}$$

We define this matrix as a "Transfer Matrix" (in a first order approximation). The main property of transfer matrices can be derived from Liouvill's theorem. This theorem states that the volume of the phase space is a constant along the path of particles. In the present calculations x and α (or y and β) may be thought of as a two-dimensional phase space because in paraxial approximation α is the ratio of the x component of the particle momentum to the total momentum. Thus areas in x-α space are proportional to phase space areas. From eq.(3-2) we can express $x_3 = x_3(x_2, \alpha_2)$ and $\alpha_3 = \alpha_3(x_2, \alpha_2)$. Then as is well known, the element of area $(dx_3 d\alpha_3)$ transforms to the area element $(dx_2 d\alpha_2)$, by means of the Jacobian determinant, written symbolically as

$$\frac{\partial(x_3, \alpha_3)}{\partial(x_2, \alpha_2)} = \begin{vmatrix} \frac{\partial x_3}{\partial x_2} & \frac{\partial \alpha_3}{\partial x_2} \\ \frac{\partial x_3}{\partial \alpha_2} & \frac{\partial \alpha_3}{\partial \alpha_2} \end{vmatrix} , \tag{3-7}$$

according to the relation

$$dx_3 d\alpha_3 = \frac{\partial(x_3, \alpha_3)}{\partial(x_2, \alpha_2)} dx_2 d\alpha_2 \quad . \tag{3-8}$$

From the conservation of phase space volume, Jacobian determinant should equal to unity. The transfer matrix is nothing but Jacobian, whose determinant equals to unity. This result is easily extended to three dimensional matrices because the third row of such matrices is invariably (0, 0, 1). The above discussions are described on the basis of the first order approximation. If we want to know the aberrations of higher order focusing, it is necessary to introduce higher order terms. The second order trajectories of the ion passing through the (x_3, y_3) plane are given by the similar relation to eq.(3-4) as follows :

$$x_3 = R_{11} x_2 + R_{12} \alpha_2 + R_{13} \delta + R_{14} x_2^2 + R_{15} x_2\alpha_2 + R_{16} x_2\delta + R_{17}\alpha_2^2 + R_{18}\alpha_2\delta + R_{19}\delta^2 + R_{110} y_2^2 + R_{111} y_2\beta_2 + R_{112} \beta_2^2 \tag{3-9a}$$

$$\alpha_3 = R_{21} x_2 + R_{22} \alpha_2 + R_{23} \delta + R_{24} x_2^2 + R_{25} x_2\alpha_2 + R_{26} x_2\delta + R_{27} \alpha_2^2 + R_{28} \alpha_2\delta + R_{29} \delta^2 + R_{210} y_2^2 + R_{211} y_2\beta_2 + R_{212} \beta_2^2 \tag{3-10a}$$

and

$$y_3 = Z_{11}\, y_2 + Z_{12}\, \beta_2 + Z_{13}\, y_2 x_2 + Z_{14}\, y_2\alpha_2 + Z_{15}\, y_2\delta$$
$$+ Z_{16}\, \beta_2 x_2 + Z_{17}\, \beta_2\alpha_2 + Z_{18}\, \beta_2\delta \qquad (3\text{-}9b)$$

$$\beta_3 = Z_{21}\, y_2 + Z_{22}\, \beta_2 + Z_{23}\, y_2 x_2 + Z_{24}\, y_2\alpha_2 + Z_{25}\, y_2\delta$$
$$+ Z_{26}\, \beta_2 x_2 + Z_{27}\, \beta_2\alpha_2 + Z_{28}\, \beta_2\delta \qquad (3\text{-}10b)$$

Though the above equations have quadratic form, they can be interpreted as linear transformations by introducing the following second order position vectors :

$$(x,\ \alpha,\ \delta,\ xx,\ x\alpha,\ x\delta,\ \alpha\alpha,\ \alpha\delta,\ \delta\delta,\ yy,\ y\beta,\ \beta\beta)$$

and

$$(y,\ \beta,\ yx,\ y\alpha,\ y\delta,\ \beta x,\ \beta\alpha,\ \beta\delta)$$

Then it is clear that the ion optical position vector of second order can also be transformed by a transfer matrix similar to eq.(3-6). In this case, however, the number of both row and column equals to twelve in radial direction and eight in axial direction ,respectively. In a third order approximation the elements of radial position vector should be

$$(x,\ \alpha,\ \delta,\ xx,\ x\alpha,\ x\delta,\ \alpha\alpha,\ \alpha\delta,\ \delta\delta,\ yy,\ y\beta,\ \beta\beta,\ xxx,\ xx\alpha,\ xx\delta,\ x\alpha\alpha,\ x\alpha\delta,$$
$$x\delta\delta,\ xyy,\ xy\beta,\ x\beta\beta,\ \alpha\alpha\alpha,\ \alpha\alpha\delta,\ \alpha\delta\delta,\ \alpha yy,\ \alpha y\beta,\ \alpha\beta\beta,\ \delta\delta\delta,\ \delta yy,\ \delta y\beta,\ \delta\beta\beta)\ .$$

The number of row and column of transfer matrix equals to 31. The detailed discussions are given in section (5.6.). It should be noticed that the elements of position vectors consist of the five independent quantities $(x,\ \alpha,\ \delta,\ y,\ \beta)$ in every case and that the determinant of transfer matrix equals to unity.

[4] The Electrostatic Potential in a Toroidal Condenser

4.1. Introduction

In order to solve the eqs.(2-30) it is necessary to determine the electric potential Φ and the magnetic vector potential A_ω of the eq.(2-29). In this section we treat the electrostatic potential in a toroidal condenser. In particle spectrometers toroidal condensers are more advantageous than the conventional cylindrical condensers, since they have the property of focusing not only in the radial but also in the axial direction. Such an electrostatic sector field is formed by radially concentric toroidal electrodes in which the separation distance is independent of the angle of deflection (see fig.2,p21). The potential in such a condenser has been determined up to terms of third order[6]. Using this result, particle trajectories can be calculated including aberrations of second order[2].

Since the electrodes are always outside of the particle beam one needs to know the electrostatic potential more precisely than what is necessary to solve the equations of motion for a particle of the beam, i.e. fourth order potential distribution. Therefore we shall derive the electrostatic potential including terms of sixth order.[4]

4.2. The Expression of the Potential

By solving the Laplaces equation under given boundary conditions, we can determine the electrostatic potential $\Phi(x,y)$.

$$\Delta\Phi(x,y) = 0 \tag{4-1}$$

In cylindrical coordinates for rotational symmetry this equation has the form :

$$\frac{1}{(\rho_0+x)}\frac{\partial}{\partial x}\left[\,(\rho_0+x)\frac{\partial\Phi}{\partial x}\,\right] + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{4-2}$$

This equation is separable in the coordinate x,y. If we set $\Phi(x,y)=R(x)\cdot Z(y)$ into eq.(4-2), after a bit of juggling we obtain

$$\frac{d^2R}{dx^2} + \frac{1}{(\rho_0+x)}\frac{dR}{dx} + k^2R = 0 \tag{4-3a}$$

$$\frac{d^2Z}{dy^2} - k^2Z = 0 \tag{4-3b}$$

We have therefore split the partial differential equation into ordinary differential equations, each for a single independent variable. The constant k is called the separation constant. The family of separated solutions of eq.(4-2) consists of the products of solutions of eq.(4-3) for all values of the parameter k. A general solution of eq.(4-2) can be expressed in terms of a linear combination of the separated solutions for various values of the parameter k.

In eq.(4-3a), since x=0 is an ordinary point, the solution R(x) can be expressed as a convergent power series around x=0.

$$R(x) = \sum_i \mu_i x^i \qquad (4\text{-}4a)$$

In eq.(4-3b) the equation for Z results in trigonometric or hyperbolic functions, depending on whether k is imaginary or real, which can be expanded in a power series in either case.

$$Z(y) = \sum_j \nu_j y^j \qquad (4\text{-}4b)$$

Then introducing eq.(4-4a, 4-4b) into $\Phi(x,y)$

$$\begin{aligned}\Phi_k(x,y) &= R(x)\ Z(y)\\ &= \sum_i \mu_i x^i \sum_j \nu_j y^j\end{aligned} \qquad (4\text{-}5)$$

In order to get the generalized solution we must sum up above solution under suitable weight

$$\Phi(x,y) = \sum_k f_k \Phi_k(x,y)$$

Rearranging this expression we get a convergent power series

$$\phi(x,y) = -E_0\rho_0 \sum \frac{a_{ij}}{i!j!}\left(\frac{x}{\rho_0}\right)^i\left(\frac{y}{\rho_0}\right)^j, \qquad (4\text{-}6)$$

where the potential of the middle equipotential surface of the toroidal condenser is assumed to be zero. E_0 describes the field strength at the middle equipotential surface, i.e. the main path x=y=0. The term $(-E_0\rho_0)$ is introduced only for simplifying the expression of a_{ij}.

Hereafter we take this expression as the potential distribution in the median plane.

4.3. The Relation between the Coefficients a_{ij} and a_{io}

Substituting eq.(4-6) into eq.(4-1) and comparing the coefficients of $x^i\ y^j$, we obtain a recursion formula ;

$$(i+1)a_{i,j+2}+a_{i+1,j+2}+(i+2)a_{i+2,j}+a_{i+3,j}=0, \tag{4-7}$$

where $a_{ij}=0$ for $i,j\leq 0$.

Using this recursion formula, the coefficients a_{ij} can be expressed as functions of a_{io}.

Since there is mirror-symmetry with respect to the median plane, all coefficients vanish when j is an odd number. The nonvanishing a_{ij} for $i+j\leq 6$ are listed below.

$$\begin{aligned}
a_{02} &= -a_{10}-a_{20},\\
a_{12} &= a_{10}-a_{20}-a_{30},\\
a_{22} &= -2a_{10}+2a_{20}-a_{30}-a_{40},\\
a_{04} &= a_{10}-a_{20}+2a_{30}+a_{40},\\
a_{32} &= 6a_{10}-6a_{20}+3a_{30}-a_{40}-a_{50},\\
a_{14} &= -3a_{10}+3a_{20}-3a_{30}+2a_{40}+a_{50},\\
a_{42} &= -24a_{10}+24a_{20}-12a_{30}+4a_{40}-a_{50}-a_{60},\\
a_{24} &= 12a_{10}-12a_{20}+9a_{30}-5a_{40}+2a_{50}+a_{60},\\
a_{06} &= -9a_{10}+9a_{20}-6a_{30}+3a_{40}-3a_{50}-a_{60}.
\end{aligned} \tag{4-8}$$

Then the next problem is to know the coefficients a_{io}.

4.4. The Boundary Conditions and the a_{io}

4.4.1. The Shape of the Electrodes

Let us assume that the electrodes are given and we want to determine the coefficients a_{io}. The two electrodes may be placed at $x=b_+$ and $x=b_-$ (see fig.3,p21). If both electrodes are perfect toroidal surface, then geometry can be described completely by their vertical radii of curvature $R(b_+)$ and $R(b_-)$. Writing b instead of b_+ and b_- we obtain

$$[x-b+R(b)]^2+y^2 = R(b)^2, \tag{4-9}$$

or

$$x = b-\frac{1}{2R(b)}y^2-\frac{1}{8R(b)^3}y^4-\frac{1}{16R(b)^5}y^6+\dots. \tag{4-10}$$

We now may determine $(d^n x/dy^n)_{x=b}$ for n=1.2.3.•••••••
from eq.(4-10)

$$\left(\frac{d^2 x}{dy^2}\right)_{x=b,\,y=0} = -\frac{1}{R(b)}, \tag{4-11a}$$

$$\left(\frac{d^4 x}{dy^4}\right)_{x=b,\,y=0} = -\frac{3}{R(b)^3}, \tag{4-11b}$$

$$\left(\frac{d^6 x}{dy^6}\right)_{x=b,\,y=0} = -\frac{45}{R(b)^5}. \tag{4-11c}$$

4.4.2. The Derivatives $d^n x/dy^n$ of the Equipotential Electrode Surfaces at x=b, y=0

The group of curved surface $\Phi(x,y)$=C(constant), changing C-value, constitutes an equipotential surface. We can assume that the equipotential surface coincides with the shape of the electrode at the neighbourhood of x=b, y=0. Keeping the relation $\Phi(x,y)$=const, x can be expanded in Taylor series around y=0.

$$x = b + \left(\frac{dx}{dy}\right)y + \frac{1}{2!}\left(\frac{d^2x}{dy^2}\right)y^2 + \frac{1}{3!}\left(\frac{d^3x}{dy^3}\right)y^3 + \cdots\cdots$$

Because of mirror-symmetry $\left(\frac{d^n x}{dy^n}\right)_{\substack{x=b\\y=0}} = 0$, for all odd n, (4-12)

$$x = b + \frac{1}{2!}\left(\frac{d^2x}{dy^2}\right)y^2 + \frac{1}{4!}\left(\frac{d^4x}{dy^4}\right)y^4 + \frac{1}{6!}\left(\frac{d^6x}{dy^6}\right)y^6 + \cdots\cdots . \tag{4-13}$$

Eq.(4-13) should coincide with eq.(4-10) up to the desired order. From mathematical consideration we can describe the derivatives $d^n x/dy^n$ as

$$\frac{dx}{dy} = -\frac{\phi_y}{\phi_x},$$

$$\frac{d^2x}{dy^2} = -\frac{\phi_{yy}}{\phi_x} + \frac{2\phi_y\phi_{xy}}{\phi_x^2} - \frac{\phi_y^2\phi_{xx}}{\phi_x^3},$$

Taking eq.(4-12) into account we obtain

$$\left(\frac{d^2x}{dy^2}\right)_{x=b,\,y=0} = -\frac{\phi_{yy}}{\phi_x}, \tag{4-14a}$$

$$\left(\frac{d^4x}{dy^4}\right)_{x=b,\,y=0} = -\frac{\phi_{yyyy}}{\phi_x} + \frac{6\phi_{yy}\phi_{xyy}}{\phi_x^2} - \frac{3\phi_{xx}\phi_{yy}^2}{\phi_x^3}, \tag{4-14b}$$

$$\begin{aligned}\left(\frac{d^6x}{dy^6}\right)_{x=b,\,y=0} = &-\frac{\phi_{yyyyyy}}{\phi_x} + \frac{15}{\phi_x^2}(\phi_{xyy}\phi_{yyyy} + \phi_{yy}\phi_{xyyyy}) \\ &- \frac{15}{\phi_x^3}(6\phi_{yy}\phi_{xyy}^2 + 3\phi_{yy}^2\phi_{xxyy} + \phi_{xx}\phi_{yy}\phi_{yyyy}) \\ &+ \frac{15}{\phi_x^4}(9\phi_{xx}\phi_{yy}^2\phi_{xyy} + \phi_{xxx}\phi_{yy}^3) - \frac{45}{\phi_x^5}\phi_{xx}^2\phi_{yy}^3.\end{aligned} \tag{4-14c}$$

Equating eqs.(4-11a) and (4-14a), (4-11b) and (4-14b), (4-11c) and (4-14c) and introducing eq.(4-6) into eqs.(4-14) we obtain

$$\begin{aligned}\left[\frac{\rho_0}{R(b)}\right] = &\ a_{02} + [a_{12} - a_{02}a_{20}]\left(\frac{b}{\rho_0}\right) + \tfrac{1}{2}[a_{22} - 2a_{12}a_{20} + (2a_{20}^2 - a_{30})a_{02}]\left(\frac{b}{\rho_0}\right)^2 \\ &+ \tfrac{1}{6}[a_{32} - 3a_{22}a_{20} + 3(2a_{20}^2 - a_{30})a_{12} + (-6a_{20}^3 + 6a_{20}a_{30} - a_{40})a_{02}]\left(\frac{b}{\rho_0}\right)^3 \\ &+ \frac{1}{24}[a_{42} - 4a_{32}a_{20} + 6(2a_{20}^2 - a_{30})a_{22} + 4(-6a_{20}^3 + 6a_{20}a_{30} - a_{40})a_{12} + (24a_{20}^4 \\ &- 36a_{20}^2a_{30} + 6a_{30}^2 + 8a_{20}a_{40} - a_{50})a_{02}]\left(\frac{b}{\rho_0}\right)^4 + \ldots,\end{aligned} \tag{4-15a}$$

$$\begin{aligned}\left[\frac{\rho_0}{R(b)}\right]^3 = &\ \tfrac{1}{3}(a_{04} - 6a_{12}a_{02} + 3a_{20}a_{02}^2) + \tfrac{1}{3}(a_{14} - a_{20}a_{04} - 6a_{22}a_{02} \\ &- 6a_{12}^2 + 18a_{12}a_{20}a_{02} + 3a_{30}a_{02}^2 - 9a_{20}^2a_{02}^2)\left(\frac{b}{\rho_0}\right) \\ &+ \tfrac{1}{6}(a_{24} - 2a_{20}a_{14} + 2a_{20}^2a_{04} - a_{30}a_{04} - 6a_{32}a_{02} - 18a_{22}a_{12} + 30a_{22}a_{02}a_{20} \\ &+ 30a_{12}^2a_{20} - 72a_{20}^2a_{12}a_{02} + 24a_{12}a_{30}a_{02} + 3a_{40}a_{02}^2 - 27a_{02}^2a_{30}a_{20} + 36a_{20}^3a_{02}^2)\left(\frac{b}{\rho_0}\right)^2 + \ldots,\end{aligned} \tag{4-15b}$$

$$\begin{aligned}\left[\frac{\rho_0}{R(b)}\right]^5 = &\ \tfrac{1}{3}\Big(\frac{a_{06}}{15} - a_{04}a_{12} - a_{02}a_{14} + 6a_{12}^2a_{02} + 3a_{22}a_{02}^2 + a_{04}a_{20}a_{02} - 9a_{12}a_{20}a_{02}^2 \\ &- a_{30}a_{02}^3 + 3a_{20}^2a_{02}^3\Big) + \ldots,\end{aligned} \tag{4-15c}$$

where b denotes either b_+ or b_- (see fig.3,p21). Introducing eqs.(4-8) into eqs.(4-15) we have five equations with five unknowns ($a_{20}, a_{30}, a_{40}, a_{50}, a_{60}$). Eqs.(4-15a) and (4-15b) are for both b_+ and b_-. Eq.(4-15c), however, represents only one equation because the right hand side of it does not contain b.

In principle we have thus determined the a_{io} if we know the shape and the position of the electrodes. In other word, we have derived the solution of the Laplace's equation that satisfies the boundary condition. If we want to determine coefficients of higher order, we can obtain each time two more coefficients by introducing the next two higher derivatives ($d^n x/dy^n$). It should be noted that because of the even higher order terms the middle equipotential surface of potential zero is not in the middle between the electrodes of potential +V and -V.

4.5. The Relation between the Coefficients a_{ij} and the shape of the Middle Equipotential Surface

For ion optical calculations it is more useful to determine the potential distribution as a function of the shape of the middle equipotential surface rather than the shape of the electrodes. Having chosen a favorable shape for the middle equipotential surface, electrodes can be calculated that produce such a middle equipotential surface.

Let us describe the quotient between the radius ρ_0 of the main path and the axial radius of curvature R(x) of any equipotential surface at y=0 as

$$\frac{\rho_0}{R(x)} = c + c'\left(\frac{x}{\rho_0}\right) + \frac{c''}{2!}\left(\frac{x}{\rho_0}\right)^2 + \frac{c'''}{3!}\left(\frac{x}{\rho_0}\right)^3 + \frac{c''''}{4!}\left(\frac{x}{\rho_0}\right)^4 + \ldots, \quad (4\text{-}16)$$

where

$$c = \left(\frac{\rho_0}{R(x)}\right)_{x=0} = \frac{\rho_0}{R_0},$$

$$c' = \rho_0\left[\frac{d}{dx}\left(\frac{\rho_0}{R(x)}\right)\right]_{x=0}, \quad (4\text{-}17)$$

$$c'' = \ \cdot\ \cdot\ \cdot\ \cdot$$

(here R_o is the axial radius of curvature of the equipotential surface x=y=o).

From ion optical requirements we can choose suitable values c and c'. Here c and c' define the axial radius of curvature R(O) of the middle equipotential surface and the change of this curvature for equipotential surfaces close to the middle one, respectively. Since the electrodes are chosen to be perfect toroidal surfaces, they form circles in the x,y plane $R(b_+)$ and $R(b_-)$. In this case we have only two parameters to determine c, c', c", •••••. Consequently, c", c''' and c'''' must be functions of c and c'.

Further we must express the coefficients a_{ij} $(i+j\leq 6)$ as functions of c and c'. This we can do in the following way. We assume that eqs.(4-15) hold for any value of b. Putting b=0 at first we get:

$$\frac{\rho_0}{R(0)} = c = a_{02}\,,$$
$$\left(\frac{\rho_0}{R(0)}\right)^3 = c^3 = \tfrac{1}{3}(a_{04}-6a_{12}a_{02}+3a_{20}a_{02}^2), \qquad (4\text{-}18)$$
$$\left(\frac{\rho_0}{R(0)}\right)^5 = c^5 = \tfrac{1}{3}\left(\frac{a_{06}}{15} - a_{04}a_{12} + \ldots\right).$$

After differentiating both sides of eqs.(4-14) after (b/ρ_0) and putting b=0, we obtain

$$\left\{\frac{d}{d(b/\rho_0)}\left[\frac{\rho_0}{R(b)}\right]\right\}_{b=0} = c' = a_{12}-a_{02}a_{20}\,,$$
$$\left\{\frac{d}{d(b/\rho_0)}\left[\frac{\rho_0}{R(b)}\right]^3\right\}_{b=0} = 3c^2c' = \tfrac{1}{3}(a_{14}-a_{20}a_{04}+\ldots). \qquad (4\text{-}19)$$

Introducing eqs.(4-8) into eqs.(4-18) and eqs.(4-19), we can express the coefficients a_{20}, a_{30}, a_{40}, a_{50}, a_{60} as function of c and c'.

$$a_{10} = 1,$$
$$a_{20} = -1-c,$$
$$a_{30} = 2+2c+c^2-c',$$
$$a_{40} = -6-5c-5c^2+2c'+6cc',$$
$$a_{50} = 24+19c+13c^2+15c^3-6c^4 -7c'-18cc'-24c^2c'+6c'^2,$$
$$a_{60} = -120-93c-60c^2-30c^3-57c^4+30c^5 +33c'+72cc'+117c^2c'+60c^3c'-18c'^2 -90cc'^2. \qquad (4\text{-}20)$$

$a_{02} = -1 - a_{20} = -1 + 1 + c$

The other coefficients a_{ij} may be obtained from eqs.(4-8) and eqs.(4-20).

4.6. The Position, Shape and Potential of the Electrodes

When the coeficients c and c' are fixed according to ion optical considerations, it must be decided where the electrodes should be placed and what shape they should have. Under the assumption that the potential of the two electrodes has symmetry around a point of the main path (x=y=0) we can obtain the position and shape of the electrodes in the following manner:

1) Knowing c and c' one determines the a_{ij} from eqs.(4-8, 4-20).
2) One sets the positive electrode at a position $b=b_+$.
3) Introducing this b_+ into eq.(4-6) one can calculate a certain value V for the corresponding potential of this electrode.
4) For some value $b=b_-$ eq.(4-6) yields the potential -V. The value of b_- may be found by using some interative method starting with the value $b_-=-b_+$.
5) Introducing these b_+ and b_- into eq.(4-15a) one can determine the corresponding axial curvatures $R(b_+)$ and $R(b_-)$ of the two electrodes.

In this manner all parameters of a toroidal condenser have been determined and the initial problem is solved.

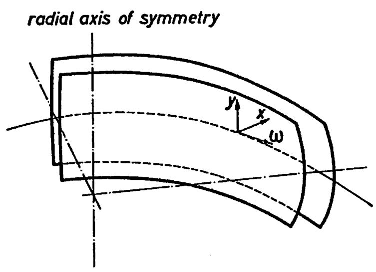


Fig. 2. Toroidal electrodes together with the radial and axial coordinates x and y.

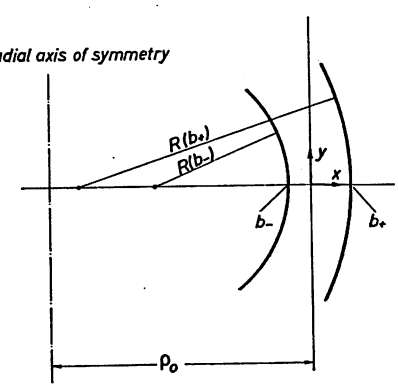


Fig. 3. The intersections of two electrodes of a toroidal condenser with a plane ω = constant.

[5] Particle Trajectories in a Toroidal Condenser

Now we will proceed to solve the differential equations by successive approximation and will arrange the results as a transfer matrix. We know two kinds of differential equations, that is, Lagrange's equations of eqs. (2-5) and Euler-Lagrange's equations of eqs.(2-30). Essentially both equations should coincide in physical meaning but the intermediate stages quite differ to each other. We have tried both ways and have got the same results.

Here we shall proceed the discussion along Euler-Lagrange's Equations. The results obtained till now are summarized in brief as follows.
The electrostatic potential $\Phi(x,y)$ is given in eq.(4-6).

$$\Phi(x,y) = -E_0\rho_0\sum \frac{a_{ij}}{i!j!}\left(\frac{x}{\rho_0}\right)^i\left(\frac{y}{\rho_0}\right)^j \tag{5-1}$$

where

$$a_{10} = 1, \qquad a_{12} = -c-c^2+c',$$
$$a_{20} = -1-c, \qquad a_{40} = -6-5c-5c^2+2c'+6cc',$$
$$a_{02} = c, \qquad a_{22} = c+4c^2-c'-6cc',$$
$$a_{30} = 2+2c+c^2-c', \qquad a_{04} = -3c^2+6cc'. \tag{5-2}$$

The c and c' describe the axial radius of curvature R(x) of equipotential surface at y=0 as :

$$\frac{\rho_0}{R(x)} = c + c'\left(\frac{x}{\rho_0}\right) + \frac{c''}{2!}\left(\frac{x}{\rho_0}\right)^2 + \cdots\cdots \tag{5-3}$$

with $c'' = -c + 2c^2 - c^3 + c' - 3cc'$

Euler-Lagrange Equations in electric field are derived from eq.(2-30).

$$\frac{d}{d\omega}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0 \tag{5-4a}$$

$$\frac{d}{d\omega}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \tag{5-4b}$$

where

$$F = \sqrt{\left[\frac{U-e\phi(x,y)}{U}\right]\left[\left(1+\frac{x}{\rho_0}\right)^2 + \left(\frac{x'}{\rho_0}\right)^2 + \left(\frac{y'}{\rho_0}\right)^2\right]} \tag{5-5}$$

$\frac{1}{2} m v^2 = eU_0$

$\frac{m_0 v_0^2}{\rho} = \frac{2eU_0}{\rho_0} = +eE_0$

$2 = \frac{+eE_0 \rho_0}{eU_0} \cong +K^2$

5.1. The Expansion of F

The potential $\Phi(x,y)$ is defined to be zero at the middle equipotential surface of the toroidal condenser. If we have ions that are not all created at the same point they may have slightly different kinetic energies U at this zero equipotential surface.

This energy may be expressed as :

$$U = U_0(1+\delta) \qquad (\delta \ll 1) \tag{5-6}$$

where U_0 is the energy of some reference particle travelling along the main path. Introducing eqs.(5-1) (5-6) into the ion optical refractive index F of eq.(5-5) we obtain

$$F(x,y,x',y',\delta) = \sqrt{\left[1+\frac{eE_0\rho_0}{U_0(1+\delta)}\sum \frac{a_{ij}}{i!j!}\left(\frac{x}{\rho_0}\right)^i\left(\frac{y}{\rho_0}\right)^j\right]\left[\left(1+\frac{x}{\rho_0}\right)^2+\left(\frac{x'}{\rho_0}\right)^2+\left(\frac{y'}{\rho_0}\right)^2\right]} \; . \tag{5-7}$$

In order to be able to solve eqs.(5-4) to a third order approximation we must expand F in a power series around $x=y=x'=y'=\delta=0$, including all terms of fourth order :

$$\begin{aligned}
F = {} & F_0+F_x\left(\frac{x}{\rho_0}\right)+F_{xx}\left(\frac{x}{\rho_0}\right)^2+F_{yy}\left(\frac{y}{\rho_0}\right)^2+F_{x\delta}\left(\frac{x}{\rho_0}\right)\delta+F_{x'x'}\left(\frac{x'}{\rho_0}\right)^2+F_{y'y'}\left(\frac{y'}{\rho_0}\right)^2+F_{xxx}\left(\frac{x}{\rho_0}\right)^3+F_{xyy}\left(\frac{x}{\rho_0}\right)\left(\frac{y}{\rho_0}\right)^2 \\
& +F_{xx\delta}\left(\frac{x}{\rho_0}\right)^2\delta+F_{yy\delta}\left(\frac{y}{\rho_0}\right)^2\delta+F_{x\delta\delta}\left(\frac{x}{\rho_0}\right)\delta^2+F_{xx'x'}\left(\frac{x}{\rho_0}\right)\left(\frac{x'}{\rho_0}\right)^2+F_{xy'y'}\left(\frac{x}{\rho_0}\right)\left(\frac{y'}{\rho_0}\right)^2+F_{xxxx}\left(\frac{x}{\rho_0}\right)^4 \\
& +F_{xxyy}\left(\frac{x}{\rho_0}\right)^2\left(\frac{y}{\rho_0}\right)^2+F_{yyyy}\left(\frac{y}{\rho_0}\right)^4+F_{xxx\delta}\left(\frac{x}{\rho_0}\right)^3\delta+F_{xyy\delta}\left(\frac{x}{\rho_0}\right)\left(\frac{y}{\rho_0}\right)^2\delta+F_{xx\delta\delta}\left(\frac{x}{\rho_0}\right)^2\delta^2+F_{yy\delta\delta}\left(\frac{y}{\rho_0}\right)^2\delta^2 \\
& +F_{x\delta\delta\delta}\left(\frac{x}{\rho_0}\right)\delta^3+F_{xxx'x'}\left(\frac{x}{\rho_0}\right)^2\left(\frac{x'}{\rho_0}\right)^2+F_{xxy'y'}\left(\frac{x}{\rho_0}\right)^2\left(\frac{y'}{\rho_0}\right)^2+F_{yyx'x'}\left(\frac{y}{\rho_0}\right)^2\left(\frac{x'}{\rho_0}\right)^2+F_{yyy'y'}\left(\frac{y}{\rho_0}\right)^2\left(\frac{y'}{\rho_0}\right)^2 \\
& +F_{xx'x'\delta}\left(\frac{x}{\rho_0}\right)\left(\frac{x'}{\rho_0}\right)^2\delta+F_{xy'y'\delta}\left(\frac{x}{\rho_0}\right)\left(\frac{y'}{\rho_0}\right)^2\delta+F_{x'x'x'x'}\left(\frac{x'}{\rho_0}\right)^4+F_{x'x'y'y'}\left(\frac{x'}{\rho_0}\right)^2\left(\frac{y'}{\rho_0}\right)^2+F_{y'y'y'y'}\left(\frac{y'}{\rho_0}\right)^4+\ldots,
\end{aligned} \tag{5-8}$$

where

$$\begin{aligned}
& F_0=1, \quad F_x=0, \quad F_{xx}=-\tfrac{1}{2}(3+a_{20}), \quad F_{yy}=-\tfrac{1}{2}a_{02}, \quad F_{x\delta}=2F_{x'x'}=2F_{y'y'}=1, \\
& F_{xxx}=-\left(1+a_{20}+\frac{a_{30}}{6}\right), \quad F_{xyy}=-(a_{02}+\tfrac{1}{2}a_{12}), \quad F_{xx\delta}=2+\tfrac{1}{2}a_{20}, \quad F_{yy\delta}=\tfrac{1}{2}a_{02}, \\
& F_{x\delta\delta}=F_{xx'x'}=F_{xy'y'}=-1, \quad F_{xxxx}= -\tfrac{1}{8}(3+a_{20})^2-\left(\frac{a_{20}}{2}+\frac{a_{30}}{3}+\frac{a_{40}}{24}\right), \\
& F_{xxyy}= -\tfrac{1}{4}(3+a_{20})a_{02}-\left(\frac{a_{02}}{2}+a_{12}+\frac{a_{22}}{4}\right), \quad F_{yyyy}= -\frac{a_{02}}{8}-\frac{a_{04}}{24}, \quad F_{xxx\delta}= \tfrac{5}{2}+\tfrac{3}{2}a_{20}+\tfrac{1}{6}a_{30}, \\
& F_{xyy\delta}=\tfrac{1}{2}(3a_{02}+a_{12}), \quad F_{xx\delta\delta}=-\tfrac{1}{2}(5+a_{20}), \quad F_{yy\delta\delta}=-\frac{a_{02}}{2}, \quad F_{x\delta\delta\delta}=1, \quad F_{xxx'x'}=F_{xxy'y'}=\tfrac{1}{4}(3-a_{20}), \\
& F_{yyx'x'}=F_{yyy'y'}=-\frac{a_{02}}{4}, \quad F_{xx'x'\delta}=F_{xy'y'\delta}=\tfrac{1}{2}, \quad F_{x'x'x'x'}=F_{y'y'y'y'}=-\tfrac{1}{8}, \quad F_{x'x'y'y'}=-\tfrac{1}{4}.
\end{aligned} \tag{5-9}$$

The relation $F_x=0$ stems from the fact that along the main path x=y=0 the centrifugal force $2U_0/\rho_0$ is balanced by the electrostatic force $-eE_0$.

This relation corresponds to the zeroth order solution of eqs.(5-4).

5.2. The Resultant Path Equations

Normally the x-focusing of a particle spectrometer is much more important than the y-focusing. Therefore one should treat eq.(5-4a) in a higher approximation than (5-4b). We solve the path equation of x-direction to a third order approximation and y-direction to a second order.
Substituting eq.(5-8) into eqs.(5-4), we obtain

$$\begin{aligned} x''-2F_{xx}x &= \rho_0\delta+3F_{xxx}x^2/\rho_0+F_{xyy}y^2/\rho_0+2F_{xx\delta}x\delta-\rho_0\delta^2+x'^2/\rho_0-y'^2/\rho_0+2xx''/\rho_0+4F_{xxxx}x^3/\rho_0^2 \\ &\quad +2F_{xxyy}xy^2/\rho_0^2+3F_{xxx\delta}x^2\delta/\rho_0+F_{xyy\delta}y^2\delta/\rho_0+2F_{xx\delta\delta}x\delta^2+\rho_0\delta^3+2F_{xxx'x'}(x^2x''-xx'^2+xy'^2)/\rho_0^2 \\ &\quad -2F_{yyx'x'}(2yx'y'+y^2x'')/\rho_0^2+\tfrac{1}{2}\delta(y'^2-x'^2-2xx'')/\rho_0+\tfrac{3}{2}x'^2x''/\rho_0^2+\tfrac{1}{2}(x''y'^2+2x'y'y'')/\rho_0^2+\ldots, \\ y''-2F_{yy}y &= 2F_{xyy}xy/\rho_0+2F_{yy\delta}y\delta+2x'y'/\rho_0+2xy''/\rho_0+\ldots, \end{aligned} \tag{5-10}$$

In order to solve eqs.(5-10) we use the method of successive approximation. In this case we can assume that the solutions are expressed in the form :

$$x(\omega) = x_{I}(\omega) + x_{II}(\omega) + x_{III}(\omega) \quad , \tag{5-11a}$$

$$y(\omega) = y_{I}(\omega) + y_{II}(\omega) \quad . \tag{5-11b}$$

Where the subscripts I, II, and III distinguish the first, the second and the third order terms, respectively. Knowing the n-th order solution we can obtain the (n+1)-th order solution if we introduce the x, y, $dx/d\omega$, $dy/d\omega$ of the n-th order solution into the right sides of eqs.(5-10) and solve the resultant differential equations in the usual manner.

We need to know $x(\omega)$ and $y(\omega)$ as well as the angles of inclination of the trajectory relative to the main path, $\alpha(\omega)$, $\beta(\omega)$ in order to describe a particle trajectory completely.
These angles are determined as

$$x'(\omega) = dx(\omega)/d\omega = (\rho_0+x)\tan\alpha(\omega) \quad , \tag{5-12a}$$

$$y'(\omega) = dy(\omega)/d\omega = (\rho_0+x)\tan\beta(\omega) \quad . \tag{5-12b}$$

Let us also distinguish here between first, second and third order terms.

$$\tan\alpha = (\tan\alpha)_{I} + (\tan\alpha)_{II} + (\tan\alpha)_{III} \quad , \tag{5-13a}$$

$$\beta = \beta_{I} + \beta_{II} \tag{5-13b}$$

where the subscripts I, II, III have the same meaning as above. To describe the particle trajectory, we shall use here $\tan\alpha$ instead of α, which is of advantage for third order calculation. In order to solve the differential equations, the initial conditions which must be known are given as :

$$x(o) = x_2, \quad x'(o) = \left(\frac{dx}{d\omega}\right)_{\omega=0} = (\rho_0 + x_2)\tan\alpha_2 \quad , \tag{5-14a}$$

$$y(o) = y_2, \quad y'(o) = \left(\frac{dy}{d\omega}\right)_{\omega=0} = (\rho_0 + x_2)\tan\beta_2 \quad . \tag{5-14b}$$

These differential equations have been already solved to a second order approximation by many authors.[2)]

Though the third order results can be obtained by extending the above method, we will introduce here a new method which is exceedingly suitable for the computer calculation. For this purpose new matrices and vectors are introduced as shown in eqs.(5-20) and (5-29). Then the right hand side of eqs.(5-10) can be interpreted as the sum of products of these matrices and vectors. Therefore their solutions will be expressed in a similar way.

5.3. The First Order Solution

Taking only the first order terms from eqs.(5-10), we obtain :

$$x_I'' + k_x^2 x_I = \rho_0 \delta \quad , \tag{5-15a}$$

$$y_I'' + k_y^2 y_I = 0 \quad , \tag{5-15b}$$

where

$$k_x = \sqrt{-2F_{xx}} = \sqrt{(2-c)} \tag{5-16a}$$

$$k_y = \sqrt{-2F_{yy}} = \sqrt{c} \quad . \tag{5-16b}$$

The solution of eqs.(5-15), $x(\omega)$, $y(\omega)$, $\tan\alpha(\omega)$, $\beta(\omega)$ are easily obtained using the initial conditions given in eqs.(5-14).

Then we have

$$x_I(\omega) = x_2 c_x + (\tan\alpha_2)\rho_0 s_x/k_x + \delta\rho_0(1-c_x)/k_x^2 \quad , \tag{5-17a}$$

$$\tan\alpha_I(\omega) = -x_2 k_x s_x/\rho_0 + (\tan\alpha_2) c_x + \delta s_x/k_x \quad , \tag{5-18a}$$

$$y_I(\omega) = y_2 c_y + \beta_2 \rho_0 s_y/k_y \quad , \tag{5-17b}$$

$$\beta_I(\omega) = -y_2 k_y s_y/\rho_0 + \beta_2 c_y \quad , \tag{5-18b}$$

with

$$s_x = \sin(k_x\omega) \quad , \quad s_y = \sin(k_y\omega) \quad ,$$
$$c_x = \cos(k_x\omega) \quad , \quad c_y = \cos(k_y\omega) \quad . \qquad (5\text{-}19)$$

The eqs.(5-17) and (5-16) represent the first order or Gaussian approximation of a particle trajectory.

5.4. The Second Order Solution

5.4.1. The Radial Trajectory

The equation which x_{II} should satisfy is obtained by substituting the first order solution eqs.(5-17), (5-18) into the right hand side of eq.(5-10) keeping only terms of second order. After rearranging systematically, this equation can then be written in the following form which consists of two vector and one matrix :

$$x''_{II} + k_x^2 x_{II} = \sum_m m \sum_{n_{II}} n_{II} \cdot G(n_{II}\,|m) \quad , \qquad (5\text{-}20a)$$

where m and n_{II} represent the components of the following vectors :

$$m = 1,\ s_x,\ c_x,\ s_x\omega,\ c_x\omega,\ s_x^2,\ s_xc_x,\ s_y^2,\ s_yc_y, \qquad (5\text{-}21a)$$

$$n_{II} = xx,\ x\alpha,\ x\delta,\ \alpha\alpha,\ \alpha\delta,\ \delta\delta,\ yy,\ y\beta,\ \beta\beta \quad , \qquad (5\text{-}22a)$$

Vector m is only conventional one which is function of field constant and aperture angle. Vector n_{II} consists of the second order components of the ion optical position vector stated in chapter [3]. It should be noticed that all matrix elements of $G(n_{II}\,|s_x\omega)$ and $G(n_{II}\,|c_x\omega)$ are equal to zero.

$G(n_{II}\,|m)$ represents the newly introduced conventional matrix, which consists of 9 x 9 elements.

	1	s_x	c_x	$s_x\omega$	$c_x\omega$	s_xs_x	s_xc_x	s_ys_y	s_yc_y
xx	G					G			
$x\alpha$							G		
$x\delta$	G		G			G			
$\alpha\alpha$	G					G			
$\alpha\delta$		G					G		
$\delta\delta$	G		G			G			
yy	G							G	
$y\beta$									G
$\beta\beta$	G							G	

where the non-zero elements are indicated by G. The explicit expressions of the non-vanishing elements of $G(n_{II}|m)$ are :

$$G(xx|1)=\frac{1}{\rho_0}(-k_x^2+q_1),\quad G(xx|s_xs_x)=\frac{1}{\rho_0}(2k_x^2-q_1),\quad G(x\alpha|s_xc_x)=-\frac{2\rho_0}{k_x}G(xx|s_xs_x),$$

$$G(x\delta|1)=-\frac{2\rho_0}{k_x^2}G(xx|1),\quad G(x\delta|c_x)=-q_8,\quad G(x\delta|s_xs_x)=-\frac{2\rho_0}{k_x^2}G(xx|s_xs_x),\quad G(\alpha\alpha|1)=\rho_0,$$

$$G(\alpha\alpha|s_xs_x)=-\frac{\rho_0^2}{k_x^2}G(xx|s_xs_x),\quad G(\alpha\delta|s_x)=\frac{\rho_0}{k_x}G(x\delta|c_x),\quad G(\alpha\delta|s_xc_x)=\frac{2\rho_0^2}{k_x^3}G(xx|s_xs_x),$$

$$G(\delta\delta|1)=\rho_0\left(\frac{1}{k_x^2}+\frac{2q_1}{k_x^4}\right),\quad G(\delta\delta|c_x)=-\frac{\rho_0}{k_x^2}G(x\delta|c_x),\quad G(\delta\delta|s_xs_x)=\frac{\rho_0^2}{k_x^4}G(xx|s_xs_x),$$

$$G(yy|1)=\frac{1}{\rho_0}(-k_y^2+q_4),\quad G(yy|s_ys_y)=-\frac{q_4}{\rho_0},\quad G(y\beta|s_yc_y)=\frac{2q_4}{k_y},\quad G(\beta\beta|1)=-\rho_0,\quad G(\beta\beta|s_ys_y)=\frac{\rho_0q_4}{k_y^2},$$

(5-23a)

where q_1 and q_4 are the abbreviations given in Table 1.

Since the right hand side of eq.(2-20a) is a function of ω through vector m, the solution $x(\omega)$ is also considered to be a series of m. Therefore we assume that

$$x_{II}(\omega) = \sum_m m \sum_{n_{II}} n_{II} \cdot H(n_{II}|m) \quad . \qquad (5\text{-}24a)$$

Introducing eq.(5-24a) into eq.(5-20a) and comparing the coefficients of equal m, we can obtain the relationships between $H(n_{II}|m)$ and $G(n_{II}|m)$. From these relations and the initial conditions given by eq.(5-14a), all the matrix elements $H(n_{II}|m)$ can be determined. The $H(n_{II}|m)$ also are elements of a 9 x 9 matrix.

	1	s_x	c_x	$s_x\omega$	$c_x\omega$	s_xs_x	s_xc_x	s_ys_y	s_yc_y
xx	H		H			H			
xα	H	H					H		
xδ	H		H	H		H			
αα	H		H			H			
αδ		H			H			H	
δδ	H		H	H		H			
yy	H							H	
yβ		H							H
ββ	H		H					H	

where the non-zero elements are indicated by H.

Explicitly the $H(n_{II}|m)$ are : [n_{II} is defined in eq.(5-22a)]

$$H(n_{II}|1) = (1/k_x^2)G(n_{II}|1) + [2/(3k_x^2)]G(n_{II}|s_xs_x) - [2k_y^2/(k_x^2q_7)]G(n_{II}|s_ys_y),$$
$$H(n_{II}|s_x) = [1/(2k_x^2)]G(n_{II}|s_x) + [1/(3k_x^2)]G(n_{II}|s_xc_x) - [k_y/(k_xq_7)]G(n_{II}|s_yc_y) + (1/k_x)\delta_{n_{II},x_2\alpha_2},$$
$$H(n_{II}|c_x) = -H(n_{II}|1), \qquad H(n_{II}|s_x\omega) = [1/(2k_x)]G(n_{II}|c_x), \qquad H(n_{II}|c_x\omega) = -[1/(2k_x)]G(n_{II}|s_x),$$
$$H(n_{II}|s_xs_x) = -[1/(3k_x^2)]G(n_{II}|s_xs_x), \qquad H(n_{II}|s_xc_x) = -[1/(3k_x^2)]G(n_{II}|s_xc_x),$$
$$H(n_{II}|s_ys_y) = (1/q_7)G(n_{II}|s_ys_y), \qquad H(n_{II}|s_yc_y) = (1/q_7)G(n_{II}|s_yc_y).$$

(5-25a)

where $\delta_{n_{II},\, x_2\alpha_2}$ is a Kronecker's δ-symbol which is always zero except for $n_{II} = x_2\alpha_2$ and q_7 is the abbreviation given in table 1.

The expression (5-25a) for $x_{II}(\omega)$ can be rearranged as

$$x_{II}(\omega) = \sum_{n_{II}} n_{II} \sum m \cdot H(n_{II}|m) = \sum_{n_{II}} n_{II} \cdot R(x|n_{II}) \qquad (5\text{-}26a)$$

In this equation the nine $R(x|n_{II})$ are the second order elements (nos. 4 to 12) of the radial transfer matrix as shown in fig.5 .p48. The n_{II} characterizes the appropriate column.

The second order approximation for $\tan\alpha$, $(\tan\alpha)_{II}$, is obtained from the relationship eq.(5-12) as

$$(\tan\alpha)_{II} = -(1/\rho_0^2)x_Ix_I' + (1/\rho_0)x_{II}' \quad .$$

Substituting eqs.(5-17a,5-26a) and rearranging,we obtain the expression

$$\begin{aligned}(\tan\alpha)_{II} &= \sum_{n_{II}} n_{II} \sum_m m \cdot H'(n_{II}|m) \\ &= \sum_{n_{II}} n_{II} \sum_m m \cdot [H_1'(n_{II}|m) + H_2'(n_{II}|m)] \\ &= \sum_{n_{II}} n_{II} \cdot R(\alpha|n_{II})\end{aligned} \qquad (5\text{-}27a)$$

where H_1' is derived from $-(1/\rho_0^2)x_Ix_I'$ and H_2' from $(1/\rho_0)x_{II}'$ and both comprise the elements of a 9 x 9 matrix.

	1	s_x	c_x	$s_x\omega$	$c_x\omega$	s_xs_x	s_xc_x	s_ys_y	s_yc_y
xx	H'						H'		
xα	H'					H	H'		
xδ		H'			H'		H'		
αα							H'		
αδ	H'		H'	H'		H'	H'		
δδ		H'			H'		H'		
yy									H'
yβ	H'							H'	
ββ									H'

All non-zero elements of this matrix are indicated by H'. Explicitly the matrix elements read :

$$H'_1(xx|s_xc_x) = k_x/\rho_0^2, \quad H'_1(x\alpha|1) = -1/\rho_0, \quad H'_1(x\alpha|s_xc_x) = 2/\rho_0, \quad H'_1(x\delta|s_x) = 1/(k_x\rho_0),$$
$$H'_1(x\delta|s_xc_x) = -2/(k_x\rho_0), \quad H'_1(\alpha\alpha|s_xc_x) = -1/k_x, \quad H'_1(\alpha\delta|1) = 1/k_x^2, \quad H'_1(\alpha\delta|c_x) = -1/k_x^2,$$
$$H'_1(\alpha\delta|s_xc_x) = -2/k_x^2, \quad H'_1(\delta\delta|s_x) = -1/k_x^3, \quad H'_1(\delta\delta|s_xc_x) = 1/k_x^3;$$
$$H'_2(n_{II}|1) = k_xH(n_{II}|s_xc_x)/\rho_0 + k_yH(n_{II}|s_yc_y)/\rho_0, \quad H'_2(n_{II}|s_x) = -k_xH(n_{II}|c_x)/\rho_0 + H(n_{II}|s_x\omega)/\rho_0,$$
$$H'_2(n_{II}|c_x) = k_xH(n_{II}|s_x)/\rho_0 + H(n_{II}|c_x\omega)/\rho_0, \quad H'_2(n_{II}|s_x\omega) = -k_xH(n_{II}|c_x\omega)/\rho_0,$$
$$H'_2(n_{II}|c_x\omega) = k_xH(n_{II}|s_x\omega)/\rho_0, \quad H'_2(n_{II}|s_xs_x) = -2k_xH(n_{II}|s_xc_x)/\rho_0, \quad H'_2(n_{II}|s_xc_x) = 2k_xH(n_{II}|s_xs_x)/\rho_0,$$
$$H'_2(n_{II}|s_ys_y) = -2k_yH(n_{II}|s_yc_y)/\rho_0, \quad H'_2(n_{II}|s_yc_y) = 2k_yH(n_{II}|s_ys_y)/\rho_0. \qquad \text{(5-28a)}$$

The nine $R(\alpha|n_{II})$ in eq.(5-27a) represent the second order elements (nos.4 to 12) of the second row of the radial transfer matrix as shown in fig.5. Here again n_{II} characterizes the appropriate column.

5.4.2. The Axial Trajectory

The solution for y equation is obtained in a way similar to x equation. From eq.(5-10b) we obtain

$$y''_{II} + k_y^2 y_{II} = \sum_m \sum_{n_{II}} m\, n_{II} \cdot G(n_{II}|m) \quad , \qquad \text{(5-20b)}$$

where in this case m and n_{II} are the components of the following vectors :

$$m = s_y,\ c_y,\ s_y\omega,\ c_y\omega,\ s_xs_y,\ s_xc_y,\ c_xs_y,\ c_xc_y, \qquad \text{(5-21b)}$$

$$n_{II} = yx,\ y\alpha,\ y\delta,\ \beta x,\ \beta\alpha,\ \beta\delta\ . \qquad \text{(5-22b)}$$

The $G(n_{II}|m)$ are now elements of 6 x 8 matrix

	s_y	c_y	$s_y\omega$	$c_y\omega$	s_xs_y	s_xc_y	c_xs_y	c_xc_y
yx					G			G
yα						G	G	
yδ		G			G			G
βx						G	G	
βα					G			G
βδ	G					G	G	

The nonvanishing elements indicated by G are :

$$G(yx|s_xs_y) = 2k_xk_y/\rho_0, \quad G(yx|c_xc_y) = q_8/\rho_0, \quad G(y\alpha|s_xc_y) = q_8/k_x, \quad G(y\alpha|c_xs_y) = -2k_y,$$
$$G(y\delta|c_y) = q_9, \quad G(y\delta|s_xs_y) = -2k_y/k_x, \quad G(y\delta|c_xc_y) = -q_8/k_x^2, \quad G(\beta x|s_xc_y) = -2k_x,$$
$$G(\beta x|c_xs_y) = q_8/k_y, \quad G(\beta\alpha|s_xs_y) = \rho_0 q_8/(k_xk_y), \quad G(\beta\alpha|c_xc_y) = 2\rho_0, \quad G(\beta\delta|s_y) = \rho_0 q_9/k_y.$$
$$G(\beta\delta|s_xc_y) = 2\rho_0/k_x, \quad G(\beta\delta|c_xs_y) = -q_8\rho_0/(k_x^2k_y),$$

(5-23b)

The solution of eq.(5-20b) can be written as :

$$y_{II}(\omega) = \sum_{m} m \sum_{n_{II}} n_{II} \cdot H(n_{II}|m) \quad . \tag{5-24b}$$

Introducing eq.(5-25b) into eq.(5-20b) and comparing the coefficients of equal m we obtain the relation between the $H(n_{II}|m)$ and $G(n_{II}|m)$. Using additionally the initial conditions of eq.(5-14b) we can determine all $H(n_{II}|m)$. The $H(n_{II}|m)$ are elements of a 6 x 8 matrix,

	s_y	c_y	$s_y\omega$	$c_y\omega$	$s_x s_y$	$s_x c_y$	$c_x s_y$	$c_x c_y$
yx		H			H			
yα	H					H	H	
yδ		H	H		H			H
βx	H					H	H	
βα		H			H			
βδ	H			H		H	H	

The non-zero elements are indicated by H. Explicitly the $H(n_{II}|m)$ are : [n_{II} is defined in eq.(5-22b)]

$$H(n_{II}|s_y) = [1/(2k_y^2)]\,G(n_{II}|s_y) + [q_6/(k_x k_y q_7)]\,G(n_{II}|s_x c_y) - (1/q_7)\,G(n_{II}|c_x s_y) + (1/k_y)\delta_{n_{II},\beta_2 x_2},$$
$$H(n_{II}|c_y) = [2k_y/(k_x q_7)]\,G(n_{II}|s_x s_y) + (1/q_7)\,G(n_{II}|c_x c_y),$$
$$H(n_{II}|s_y\omega) = [1/(2k_y)]\,G(n_{II}|c_y), \qquad H(n_{II}|c_y\omega) = -[1/(2k_y)]\,G(n_{II}|s_y\ ),$$
$$H(n_{II}|s_x s_y) = -(1/q_7)\,G(n_{II}|s_x s_y) - [2k_y/(k_x q_7)]\,G(n_{II}|c_x c_y),$$
$$H(n_{II}|s_x c_y) = -(1/q_7)\,G(n_{II}|s_x c_y) + [2k_y/(k_x q_7)]\,G(n_{II}|c_x s_y),$$
$$H(n_{II}|c_x s_y) = [2k_y/(k_x q_7)]\,G(n_{II}|s_x c_y) - (1/q_7)\,G(n_{II}|c_x s_y), \qquad H(n_{II}|c_x c_y) = -H(n_{II}|c_y), \tag{5-25b}$$

where q_6 and q_7 are the abbreviations in table 1. The expansion (5-24b) of $y_{II}(\omega)$ similar to eq.(5-26a) can be arranged as

$$y_{II}(\omega) = \sum_{n_{II}} n_{II} \sum_{m} m \cdot H(n_{II}|m) = \sum_{n_{II}} n_{II} \cdot R(y|n_{II}) . \tag{5-26b}$$

The 7 $R(y|n_{II})$ are the second order elements (nos.3 to 8) of axial transfer matrix.

Similar to eq.(5-28a) β_{II} is written as

$$\beta_{II} = \sum_{n_{II}} n_{II} \sum_{m} m \cdot [H_1'(n_{II}|m) + H_2'(n_{II}|m)] = \sum_{n_{II}} n_{II} \cdot R(\beta|n_{II}) \tag{5-27b}$$

The H_1' and H_2' comprise the elements of a 6 x 8 matrix

	s_y	c_y	$s_y\omega$	$c_y\omega$	$s_x s_y$	$s_x c_y$	$c_x s_y$	$c_x c_y$
yx						H'	H'	
yα					H'			H'
yδ				H'		H'	H'	
βx					H'			H'
βα						H'	H'	
βδ		H'	H'		H'			H'

All non-zero elements of this matrix are indicated by H'. Explicitly these elements read :

$$H'_1(yx|c_xs_y) = k_y/\rho_0^2, \quad H'_1(y\alpha|s_xs_y) = k_y/(\rho_0 k_x), \quad H'_1(y\delta|s_y) = k_y/(\rho_0 k_x^2), \quad H'_1(y\delta|c_xs_y) = -k_y/(\rho_0 k_x^2),$$
$$H'_1(\beta x|c_xc_y) = -1/\rho_0, \quad H'_1(\beta\alpha|s_xc_y) = -1/k_x, \quad H'_1(\beta\delta|c_y) = -1/k_x^2, \quad H'_1(\beta\delta|c_xc_y) = 1/k_x^2;$$
$$H'_2(n_{II}|s_y) = -k_yH(n_{II}|c_y)/\rho_0 + H(n_{II}|s_y\omega)/\rho_0, \quad H'_2(n_{II}|c_y) = k_yH(n_{II}|s_y)/\rho_0 + H(n_{II}|c_y\omega)/\rho_0,$$
$$H'_2(n_{II}|s_y\omega) = -k_yH(n_{II}|c_y\omega)/\rho_0, \quad H'_2(n_{II}|c_y\omega) = k_yH(n_{II}|s_y\omega)/\rho_0,$$
$$H'_2(n_{II}|s_xs_y) = -k_yH(n_{II}|s_xc_y)/\rho_0 - k_xH(n_{II}|c_xs_y)/\rho_0, \quad H'_2(n_{II}|s_xc_y) = k_yH(n_{II}|s_xs_y)/\rho_0 - k_xH(n_{II}|c_xc_y)/\rho_0,$$
$$H'_2(n_{II}|c_xs_y) = k_xH(n_{II}|s_xs_y)/\rho_0 - k_yH(n_{II}|c_xc_y)/\rho_0, \quad H'_2(n_{II}|c_xc_y) = k_xH(n_{II}|s_xc_y)/\rho_0 + k_yH(n_{II}|c_xs_y)/\rho_0.$$

(5-28b)

The coefficients $R(\beta|n_{II})$ in eq.(5-27b) represent the second order elements (nos.3 to 8) of the second row of the axial transfer matrix shown in fig.4.

	y	β	yx	yα	yδ	βx	βα	βδ
y	R	R	R	R	R	R	R	R
β	R	R	R	R	R	R	R	R
yx			R	R	R	R	R	R
yα			R	R	R	R	R	R
yδ			R	R	R	R	R	R
βx			R	R	R	R	R	R
βα			R	R	R	R	R	R
βδ			R	R	R	R	R	R

Fig.4. The axial transfer matrix of second order. The elements of the first and the second row are given by eqs.(5-17b), (5-18b), (5-26b) and (5-27b). The other matrix elements are given by eq.(5-42).

5.5. The Radial Third Order Solution

The differential equation for x_{III} is obtained by substituting the first order solutions (5-17a, 5-18a) and the second order solutions (5-24a, 5-24b) into the right hand side of eq.(5-10) and taking only the third order terms. The equation is written as

$$x''_{III} + k_x^{\,2} x_{III} = \sum_m m \sum_{n_{III}} n_{III} \cdot G(n_{III}|m) \quad , \tag{5-29}$$

where m and n_{III} now represent the components of the following vectors :

$$m = 1,\ \omega,\ s_x,\ c_x,\ s_x\omega,\ c_x\omega,\ s_x^{\,2},\ s_xc_x,\ s_y^{\,2},\ s_yc_y,\ s_x\omega^2,\ c_x\omega^2,\ s_x^{\,2}\omega,\ s_xc_x\omega,$$
$$s_y^{\,2}\omega,\ s_yc_y\omega,\ s_x^{\,3},\ s_x^{\,2}c_x,\ s_xs_y^{\,2},\ s_xs_yc_y,\ c_xs_y^{\,2},\ c_xs_yc_y, \tag{5-30}$$

$$n_{III} = xxx,\ xx\alpha,\ xx\delta,\ x\alpha\alpha,\ x\alpha\delta,\ x\delta\delta,\ xyy,\ xy\beta,\ x\beta\beta,\ \alpha\alpha\alpha,\ \alpha\alpha\delta,\ \alpha\delta\delta,\ \alpha yy,$$
$$\alpha y\beta,\ \alpha\beta\beta,\ \delta\delta\delta,\ \delta yy,\ \delta y\beta,\ \delta\beta\beta\ . \tag{5-31}$$

The $G(n_{III}|m)$ are elements of a 19 x 22 matrix

	1	ω	s_x	c_x	$s_x\omega$	$c_x\omega$	s_xs_x	s_xc_x	s_ys_y	s_yc_y	$s_x\omega\omega$	$c_x\omega\omega$	$s_xs_x\omega$	$s_xc_x\omega$	$s_ys_y\omega$	$s_yc_y\omega$	$s_xs_xs_x$	$s_xs_xc_x$	$s_xs_ys_y$	$s_xs_yc_y$	$c_xs_ys_y$	$c_xs_yc_y$
xxx	G			G			G											G				
xxα			G					G									G					
xxδ	G			G			G							G				G				
xαα	G			G				G										G				
xαδ		G	G					G					G				G					
xδδ	G			G	G		G							G				G				
xyy	G			G			G		G											G	G	
xyβ			G					G		G									G			G
xββ	G			G				G	G											G	G	
ααα			G					G									G					
ααδ	G			G				G						G				G				
αδδ		G	G			G		G					G				G					
αyy			G					G		G									G			G
αyβ	G			G			G		G											G	G	
αββ			G					G		G									G			G
δδδ	G			G	G		G							G				G				
δyy	G			G				G	G							G				G	G	
δyβ		G	G					G		G					G				G			G
δββ	G			G			G		G							G				G	G	

,

where all non-zero elements are indicated by G. Explicitly these $G(n_{III}|m)$ are :

$$G(xxx|1) = \frac{2}{3\rho_0^2}\left(k_x^2 - \frac{q_1^2}{k_x^2}\right), \quad G(xxx|c_x) = -\frac{1}{\rho_0^2}\left(k_x^4 + k_x^2 - \frac{2q_1}{3} - \frac{2q_1^2}{3k_x^2} - q_2\right),$$

$$G(xxx|s_xs_x)=\frac{2}{3\rho_0^2}\left(-2k_x^2-q_1+\frac{q_1^2}{k_x^2}\right),\qquad G(xxx|s_xs_xc_x)=\frac{1}{\rho_0^2}\left(\frac{10k_x^2}{3}-4q_1+\frac{2q_1^2}{3k_x^2}-q_2\right),$$

$$G(xx\alpha|s_x)=\frac{1}{\rho_0}\left(-k_x^3-\frac{23k_x}{3}+\frac{26q_1}{3k_x}-\frac{2q_1^2}{3k_x^3}+\frac{3q_2}{k_x}\right),\qquad G(xx\alpha|s_xc_x)=\frac{2}{3\rho_0}\left(4k_x-\frac{4q_1}{k_x}+\frac{q_1^2}{k_x^3}\right),$$

$$G(xx\alpha|s_xs_xs_x)=\frac{3\rho_0}{k_x}G(xxx|s_xs_xc_x),\qquad G(xx\delta|1)=\frac{1}{\rho_0}\left(\frac{4k_x^2}{3}+\frac{1}{3}-\frac{2q_1}{3}+\frac{3q_1}{k_x^2}+\frac{8q_1^2}{3k_x^4}+\frac{3q_2}{k_x^2}\right),$$

$$G(xx\delta|c_x)=\frac{1}{\rho_0}(4k_x^2+3-q_1)-G(xx\delta|1),\qquad G(xx\delta|s_xs_x)=\frac{1}{\rho_0}\left(-\frac{8k_x^2}{3}-\frac{2}{3}+\frac{4q_1}{3}-\frac{3q_1}{k_x^2}-\frac{4q_1^2}{3k_x^4}-\frac{3q_2}{k_x^2}\right),$$

$$G(xx\delta|s_xc_x\omega)=\frac{1}{\rho_0}\left(-2k_x^3-2k_x+k_xq_1-\frac{3q_1}{k_x}+\frac{2q_1^2}{k_x^3}\right),\qquad G(xx\delta|s_xs_xc_x)=-\frac{3\rho_0}{k_x^2}G(xxx|s_xs_xc_x),$$

$$G(x\alpha\alpha|1)=\frac{2}{3}\left(-2+\frac{5q_1}{k_x^2}-\frac{2q_1^2}{k_x^4}\right),\qquad G(x\alpha\alpha|c_x)=-k_x^2+1-G(x\alpha\alpha|1),\qquad G(x\alpha\alpha|s_xc_x)=-2G(x\alpha\alpha|1),$$

$$G(x\alpha\alpha|s_xs_xc_x)=-\frac{3\rho_0^2}{k_x^2}G(xxx|s_xs_xc_x),\qquad G(x\alpha\delta|\omega)=-\frac{\rho_0}{k_x}G(xx\delta|s_xc_x\omega),$$

$$G(x\alpha\delta|s_x)=\left(\frac{5k_x}{3}+\frac{15}{k_x}+\frac{2q_1}{3k_x}-\frac{52q_1}{3k_x^3}+\frac{8q_1^2}{3k_x^5}-\frac{6q_2}{k_x^3}\right),\qquad G(x\alpha\delta|s_xc_x)=\left(\frac{10k_x}{3}-\frac{2}{k_x}-\frac{5q_1}{3k_x}+\frac{43q_1}{3k_x^3}-\frac{2q_1^2}{3k_x^5}+\frac{6q_2}{k_x^3}\right),$$

$$G(x\alpha\delta|s_xs_x\omega)=\frac{2\rho_0}{k_x}G(xx\delta|s_xc_x\omega),\qquad G(x\alpha\delta|s_xs_xs_x)=-\frac{6\rho_0^2}{k_x^3}G(xxx|s_xs_xc_x),$$

$$G(x\delta\delta|1)=\left(-\frac{8}{3}+\frac{4}{3k_x^2}+\frac{4q_1}{3k_x^2}-\frac{8q_1}{k_x^4}-\frac{16q_1^2}{3k_x^6}-\frac{6q_2}{k_x^4}\right),\qquad G(x\delta\delta|c_x)=-k_x^2-\frac{15}{3}-G(x\delta\delta|1),$$

$$G(x\delta\delta|s_x\omega)=\left(\frac{k_x^3}{2}+k_x+\frac{1}{2k_x}+\frac{2q_1}{k_x}+\frac{2q_1}{k_x^3}+\frac{2q_1^2}{k_x^5}\right),\qquad G(x\delta\delta|s_xs_x)=\left(\frac{16}{3}-\frac{8}{3k_x^2}-\frac{8q_1}{3k_x^2}+\frac{8q_1}{k_x^4}+\frac{8q_1^2}{3k_x^6}+\frac{6q_2}{k_x^4}\right),$$

$$G(x\delta\delta|s_xc_x\omega)=-\frac{2\rho_0}{k_x^2}G(xx\delta|s_xc_x\omega),\qquad G(x\delta\delta|s_xs_xc_x)=\frac{3\rho_0^2}{k_x^4}G(xxx|s_xs_xc_x),$$

$$G(xyy|1)=\frac{1}{\rho_0^2}\left(-5k_x^2+4-\frac{2q_1q_5}{k_x^2}-\frac{2q_1q_4q_6}{k_x^2q_7}+\frac{4q_4^2}{q_7}\right),$$

$$G(xyy|c_x)=\frac{1}{\rho_0^2}\left(-(1+k_x^2)k_y^2-\frac{2k_y^2q_1}{k_x^2}+\frac{2q_1q_4q_6}{k_x^2q_7}-\frac{4q_4^2}{q_7}+q_3\right),$$

$$G(xyy|s_xs_x)=\frac{1}{\rho_0^2}\left(4k_y^2-\frac{2k_y^2q_1}{k_x^2}-\frac{4q_4q_6}{q_7}+\frac{2q_1q_4q_6}{k_x^2q_1}\right),\qquad G(xyy|s_ys_y)=-\frac{4q_4^2}{\rho_0^2q_7},$$

$$G(xyy|s_xs_yc_y)=\frac{8k_y}{\rho_0^2q_7}\left(2k_xq_4+\frac{q_1q_4}{k_x}\right);\qquad G(xyy|c_xs_ys_y)=\frac{1}{\rho_0^2}\left(-\frac{2q_1q_4}{q_7}+\frac{4q_4^2}{q_7}-q_3\right),$$

$$G(xy\beta|s_x)=-\frac{\rho_0}{k_y}G(xyy|s_xs_yc_y),\qquad G(xy\beta|s_xc_x)=\frac{4}{\rho_0}\left(\frac{2k_xq_4}{q_7}-\frac{q_1q_4}{k_xq_7}\right),\qquad G(xy\beta|s_yc_y)=-\frac{2q_4}{\rho_0k_y},$$

$$G(xy\beta|s_xs_ys_y)=\frac{2\rho_0}{k_y}G(xyy|s_xs_yc_y),\qquad G(xy\beta|c_xs_yc_y)=-\frac{2\rho_0}{k_y}G(xyy|c_xs_ys_y),\qquad G(x\beta\beta|1)=\left(\frac{2q_1}{k_x^2}+\frac{4q_1q_4}{k_x^2q_7}\right),$$

$$G(x\beta\beta|c_x)=-k_x^2-1-G(x\beta\beta|1),\qquad G(x\beta\beta|s_xs_x)=\left(4-\frac{2q_1}{k_x^2}+\frac{8q_4}{q_7}-\frac{4q_1q_4}{k_x^2q_7}\right),$$

$$G(x\beta\beta|s_ys_y)=\left(\frac{4k_x^2q_4}{k_y^2q_7}+\frac{4q_1q_4}{k_y^2q_7}\right),\qquad G(x\beta\beta|s_xs_yc_y)=-\frac{\rho_0^2}{k_y^2}G(xyy|s_xs_yc_y),\qquad G(x\beta\beta|c_xs_ys_y)=-\frac{\rho_0^2}{k_y^2}G(xyy|c_xs_ys_y),$$

$$G(\alpha\alpha\alpha|s_x)=\frac{\rho_0}{k_x}G(x\alpha\alpha|c_x),\qquad G(\alpha\alpha\alpha|s_xc_x)=\frac{\rho_0}{k_x}G(x\alpha\alpha|1),\qquad G(\alpha\alpha\alpha|s_xs_xs_x)=-\frac{\rho_0^3}{k_x^3}G(xxx|s_xs_xc_x),$$

$$G(\alpha\alpha\delta|1)=\rho_0\left(-\frac{1}{3}+\frac{2}{k_x^2}+\frac{2q_1}{3k_x^2}-\frac{10q_1}{3k_x^4}+\frac{8q_1^2}{3k_x^6}\right),\qquad G(\alpha\alpha\delta|c_x)=\rho_0-G(\alpha\alpha\delta|1),$$

$$G(\alpha\alpha\delta|s_xs_x)=\rho_0\left(\frac{2}{3}-\frac{4}{k_x^2}-\frac{q_1}{3k_x^2}+\frac{32q_1}{3k_x^4}-\frac{4q_1^2}{3k_x^6}+\frac{3q_2}{k_x^4}\right),\qquad G(\alpha\alpha\delta|s_xc_x\omega)=-\frac{\rho_0^2}{k_x^2}G(xx\delta|s_xc_x\omega),$$

$$G(\alpha\alpha\delta|s_xs_xc_x)=\frac{3\rho_0^3}{k_x^4}G(xxx|s_xs_xc_x),\qquad G(\alpha\delta\delta|\omega)=\frac{\rho_0^2}{k_x^3}G(xx\delta|s_xc_x\omega),$$

$$G(\alpha\delta\delta|s_x)=\rho_0\left(-\frac{k_x}{2}+\frac{4}{3k_x}-\frac{2g}{6k_x^3}-\frac{2q_1}{3k_x^3}+\frac{50q_1}{3k_x^5}+\frac{2q_1^2}{k_x^7}+\frac{6q_2}{k_x^5}\right),\qquad G(\alpha\delta\delta|c_x\omega)=-\frac{\rho_0}{k_x}G(x\delta\delta|s_x\omega),$$

$$G(\alpha\delta\delta|s_xc_x)=\rho_0\left(-\frac{10}{3k_x}+\frac{22}{3k_x^3}+\frac{5q_1}{3k_x^3}-\frac{35q_1}{3k_x^5}-\frac{2q_1^2}{k_x^7}-\frac{6q_2}{k_x^5}\right),\qquad G(\alpha\delta\delta|s_xs_x\omega)=-\frac{2\rho_0^2}{k_x^3}G(xx\delta|s_xc_x\omega),$$

$$G(\alpha\delta\delta|s_xs_xs_x)=\frac{3\rho_0^3}{k_x^5}G(xxx|s_xs_xc_x),\qquad G(\alpha yy|s_x)=\frac{\rho_0}{k_x}G(xyy|c_x),\qquad G(\alpha yy|s_xc_x)=-\frac{\rho_0}{k_x}G(xyy|s_xs_x),$$

$$G(\alpha yy|s_yc_y)=\frac{1}{\rho_0}\left(\frac{8k_yq_4}{q_7}+\frac{8k_yq_1q_4}{k_x^2q_7}+\frac{4q_4^2}{k_yq_7}\right),\qquad G(\alpha yy|s_xs_ys_y)=\frac{\rho_0}{k_x}G(xyy|c_xs_ys_y),$$

$$G(\alpha yy|c_xs_yc_y)=-\frac{\rho_0}{k_x}G(xyy|s_xs_yc_y),\qquad G(\alpha y\beta|1)=-\frac{\rho_0^2}{k_xk_y}G(xyy|s_xs_yc_y),\qquad G(\alpha y\beta|c_x)=\frac{\rho_0^2}{k_xk_y}G(xyy|s_xs_yc_y),$$

$$G(\alpha y\beta|s_xs_x)=\frac{\rho_0}{k_x}G(xy\beta|s_xc_x),\qquad G(\alpha y\beta|s_ys_y)=\left(-\frac{6q_4}{k_y^2}-\frac{4q_1q_4}{k_x^2k_y^2}\right),\qquad G(\alpha y\beta|s_xs_yc_y)=-\frac{2\rho_0^2}{k_xk_y}G(xyy|c_xs_ys_y),$$

$$G(\alpha y\beta|c_xs_ys_y)=-\frac{2\rho_0^2}{k_xk_y}G(xyy|s_xs_yc_y),\qquad G(\alpha\beta\beta|s_x)=-\rho_0\left(k_x+\frac{1}{k_x}+\frac{2q_1}{k_x^3}+\frac{4q_1q_4}{k_x^3q_7}\right),$$

$$G(\alpha\beta\beta|s_xc_x)=-\frac{\rho_0}{k_x}G(x\beta\beta|s_xs_x),\qquad G(\alpha\beta\beta|s_yc_y)=-\frac{2k_y\rho_0}{k_x^2}G(x\beta\beta|s_ys_y),\qquad G(\alpha\beta\beta|s_xs_ys_y)=-\frac{\rho_0^3}{k_xk_y^2}G(xyy|c_xs_ys_y),$$

$$G(\alpha\beta\beta|c_xs_yc_y)=\frac{\rho_0^3}{k_xk_y^2}G(xyy|s_xs_yc_y),\qquad G(\delta\delta\delta|1)=\rho_0\left(\frac{4}{3k_x^2}+\frac{7}{3k_x^4}-\frac{2q_1}{3k_x^4}+\frac{32q_1}{3k_x^6}+\frac{16q_1^2}{3k_x^8}+\frac{4q_2}{k_x^6}\right),$$

$$G(\delta\delta\delta|c_x)=\rho_0-G(\delta\delta\delta|1),\qquad G(\delta\delta\delta|s_x\omega)=-\frac{\rho_0}{k_x^2}G(x\delta\delta|s_x\omega),$$

$$G(\delta\delta\delta|s_xs_x)=\rho_0\left(-\frac{8}{3k_x^2}+\frac{14}{3k_x^4}+\frac{4q_1}{3k_x^4}-\frac{13q_1}{3k_x^6}-\frac{2q_1^2}{k_x^8}-\frac{3q_2}{k_x^6}\right),\qquad G(\delta\delta\delta|s_xc_x\omega)=\frac{\rho_0^2}{k_x^4}G(xx\delta|s_xc_x\omega),$$

$$G(\delta\delta\delta|s_xs_xc_x)=-\frac{\rho_0^3}{k_x^6}G(xxx|s_xs_xc_x),$$

$$G(\delta yy|1)=\frac{1}{\rho_0}\left[-\frac{2k_y^2}{k_x^2}-\frac{4k_y^2q_1}{k_x^4}+\left(\frac{2k_y^2}{q_1}+\frac{7k_x^2-10k_y^2}{k_x^2q_1}\right)q_4+\frac{8q_1q_4q_5}{k_x^4q_7}+\frac{q_3}{k_x^2}\right],\qquad G(\delta yy|c_x)=\frac{2k_y^2}{\rho_0}-G(\delta yy|1),$$

$$G(\delta yy|s_xs_x)=-\frac{\rho_0}{k_x^2}G(xyy|s_xs_x),\qquad G(\delta yy|s_ys_y)=\frac{1}{\rho_0}\left(-\frac{7q_4}{q_7}-\frac{4k_y^4q_4}{k_x^2q_7}-\frac{6q_1q_4}{k_x^2q_7}-\frac{q_3}{k_x^2}\right),$$

$$G(\delta yy|s_yc_y\omega)=\frac{1}{\rho_0}\left(-\frac{q_4}{k_y}-\frac{k_x^2q_4}{k_y}-\frac{2q_1q_4}{k_x^2k_y}\right),\qquad G(\delta yy|s_xs_yc_y)=-\frac{\rho_0}{k_x^2}G(xyy|s_xs_yc_y),$$

$$G(\delta yy|c_xs_ys_y)=-\frac{\rho_0}{k_x^2}G(xyy|c_xs_ys_y),\qquad G(\delta y\beta|\omega)=-\frac{\rho_0}{k_y}G(\delta yy|s_yc_y\omega),\qquad G(\delta y\beta|s_x)=\left[\frac{2(7-k_x^2)q_4}{k_xq_1}+\frac{4q_1q_4}{k_x^3q_7}\right],$$

$$G(\delta y\beta|s_xc_x)=-\frac{\rho_0}{k_x^2}G(xy\beta|s_xc_x),\qquad G(\delta y\beta|s_yc_y)=\left[\left(\frac{4}{k_x^2k_y}+\frac{2+2k_x^2}{k_yq_7}-\frac{3+k_y^2}{k_y^3}\right)q_4+\frac{2(-k_x^2+6k_y^2)q_1q_4}{k_x^2k_y^3q_7}+\frac{2q_3}{k_x^2k_y}\right],$$

$$G(\delta y\beta|s_ys_y\omega)=\frac{2\rho_0}{k_y}G(\delta yy|s_yc_y\omega),\qquad G(\delta y\beta|s_xs_ys_y)=-\frac{2\rho_0^2}{k_x^2k_y}G(xyy|s_xs_yc_y),\qquad G(\delta y\beta|c_xs_yc_y)=\frac{2\rho_0^2}{k_x^2k_y}G(xyy|c_xs_ys_y),$$

$$G(\delta\beta\beta|1)=-\rho_0\left[1+\frac{2}{k_x^2}+\frac{4q_1}{k_x^4}+\frac{2(1+k_x^2)q_4}{k_x^2q_7}+\frac{8q_1q_4}{k_x^4q_7}\right]\qquad G(\delta\beta\beta|c_x)=\rho_0-G(\delta\beta\beta|1),$$

$$G(\delta\beta\beta|s_xs_x)=-\frac{\rho_0}{k_x^2}G(x\beta\beta|s_xs_x),$$

$$G(\delta\beta\beta|s_ys_y)=\rho_0\left[\left(\frac{1+k_x^2}{k_y^2q_7}-\frac{2k_x^2+8k_y^2}{k_x^2k_y^2q_7}-\frac{1+k_x^2}{k_y^4}-\frac{1}{k_y^2}\right)q_4-\frac{2(k_x^2-3k_y^2)q_1q_4}{k_x^2k_y^4q_7}+\frac{q_3}{k_x^2k_y^2}\right],$$

$$G(\delta\beta\beta|s_yc_y\omega)=-\frac{\rho_0^2}{k_y^2}G(\delta yy|s_yc_y\omega),\qquad G(\delta\beta\beta|s_xs_yc_y)=\frac{\rho_0^3}{k_x^2k_y^2}G(xyy|s_xs_yc_y),\qquad G(\delta\beta\beta|c_xs_ys_y)=\frac{\rho_0^3}{k_x^2k_y^2}G(xyy|c_xs_ys_y)\,.$$

(5-32)

where q_1, q_2, q_3, •••••, q_7 are the abbreviation given in Table 1. To solve eq.(5-29) we will use a method similar to the second order solution. Since the right hand side of eq.(5-20a) is a function of ω through vector m, $x_{III}(\omega)$ is also considered to be a series of m. Therefore we assume that

$$x_{III}(\omega) = \sum_m m \sum_{n_{III}} n_{III}\cdot H(n_{III}|m)\ . \qquad (5\text{-}33)$$

Introducing eq.(5-33) into eq.(5-29) and comparing the coefficients of equal m, we can obtain the relationships between $H(n_{III}|m)$ and $G(n_{III}|m)$. From these relations and the initial conditions, all the matrix elements $H(n_{III}|m)$ are determined. The $H(n_{III}|m)$ are elements of a 19 x 22 matrix,

	1	ω	s_x	c_x	$s_x\omega$	$c_x\omega$	s_xs_x	s_xc_x	s_ys_y	s_yc_y	$s_x\omega\omega$	$c_x\omega\omega$	$s_xs_x\omega$	$s_xc_x\omega$	$s_ys_y\omega$	$s_yc_y\omega$	$s_xs_xs_x$	$s_xs_xc_x$	$s_xs_ys_y$	$s_xs_yc_y$	c_xs_y	$c_xs_yc_y$
xxx	H			H	H		H											H				
xxα			H			H		H									H					
xxδ	H			H	H		H							H				H				
xαα	H		H	H	H			H										H				
xαδ		H	H			H		H					H				H					
δδδ	H			H	H		H					H		H				H				
xyy	H			H	H		H		H											H	H	
xyβ			H			H		H		H									H			H
xββ	H		H	H	H			H	H											H	H	
ααα			H					H									H					
ααδ	H		H	H	H		H	H						H				H				
αδδ		H	H			H		H			H		H				H					
αyy			H			H		H		H									H			H
αyβ	H			H	H		H		H											H	H	
αββ			H			H		H		H									H			H
δδδ	H			H	H	H	H					H		H				H				
δyy	H		H	H				H	H							H				H	H	
δyβ		H	H			H		H		H					H				H			H
δββ	H			H	H		H		H							H				H	H	

where all non-zero elements are indicated by H.

Explicitly these $H(n_{III}|m)$ are :

$$H(n_{III}|1)=\frac{1}{k_x^2}G(n_{III}|1)+\frac{2}{3k_x^2}G(n_{III}|s_xs_x)-\frac{2k_y^2}{k_x^2q_7}G(n_{III}|s_ys_y)-\frac{2}{9k_x^3}G(n_{III}|s_xc_x\omega)-\frac{2k_y}{q_7^2}G(n_{III}|s_yc_y\omega),$$

$$H(n_{III}|\omega)=\frac{1}{k_x^2}G(n_{III}|\omega)+\frac{2}{3k_x^2}G(n_{III}|s_xs_x\omega)-\frac{2k_y^2}{k_x^2q_7}G(n_{III}|s_ys_y\omega),$$

$$H(n_{III}|s_x)=-\frac{1}{k_x^3}G(n_{III}|\omega)+\frac{1}{2k_x^2}G(n_{III}|s_x)-\frac{1}{4k_x^3}G(n_{III}|c_x\omega)+\frac{1}{3k_x^2}G(n_{III}|s_xc_x)-\frac{k_y}{k_xq_7}G(n_{III}|s_yc_y)-\frac{2}{qk_x^3}G(n_{III}|s_xs_x\omega)$$
$$+\frac{2k_y^2(2k_x^2+q_7)}{k_x^3q_7^2}G(n_{III}|s_ys_y\omega)+\frac{3}{8k_x^2}G(n_{III}|s_xs_xs_x)-\frac{k_y^2}{4k_x^2q_5}G(n_{III}|s_xs_ys_y)-\frac{k_y}{4k_xq_5}G(n_{III}|c_xs_yc_y),$$

$$H(n_{III}|c_x)=-H(n_{III}|1),\qquad H(n_{III}|s_x\omega)=\frac{1}{2k_x}G(n_{III}|c_x)+\frac{1}{4k_x^2}G(n_{III}|s_x\omega)+\frac{1}{8k_x}G(n_{III}|s_xs_xc_x)+\frac{1}{4k_x}G(n_{III}|c_xs_ys_y),$$

$$H(n_{III}|c_x\omega)=-\frac{1}{2k_x}G(n_{III}|s_x)+\frac{1}{4k_x^2}G(n_{III}|c_x\omega)-\frac{3}{8k_x}G(n_{III}|s_xs_xs_x)-\frac{1}{4k_x}G(n_{III}|s_xs_ys_y),$$

$$H(n_{III}|s_xs_x)=-\frac{1}{3k_x^2}G(n_{III}|s_xs_x)+\frac{4}{qk_x^3}G(n_{III}|s_xc_x\omega),\qquad H(n_{III}|s_xc_x)=-\frac{1}{3k_x^2}G(n_{III}|s_xc_x)-\frac{4}{qk_x^3}G(n_{III}|s_xs_x\omega),$$

$$H(n_{III}|s_ys_y)=\frac{1}{q_7}G(n_{III}|s_ys_y)+\frac{4k_y}{q_7^2}G(n_{III}|s_yc_y\omega),\qquad H(n_{III}|s_yc_y)=\frac{1}{q_7}G(n_{III}|s_yc_y)-\frac{4k_y}{q_7^2}G(n_{III}|s_ys_y\omega),$$

$$H(n_{III}|s_x\omega\omega)=\frac{1}{4k_x}G(n_{III}|c_x\omega),\qquad H(n_{III}|c_x\omega\omega)=-\frac{1}{4k_x}G(n_{III}|s_x\omega),\qquad H(n_{III}|s_xs_x\omega)=-\frac{1}{3k_x^2}G(n_{III}|s_xs_x\omega),$$

$$H(n_{III}|s_xc_x\omega)=-\frac{1}{3k_x^2}G(n_{III}|s_xc_x\omega),\qquad H(n_{III}|s_ys_y\omega)=\frac{1}{q_7}G(n_{III}|s_ys_y\omega),\qquad H(n_{III}|s_yc_y\omega)=\frac{1}{q_7}G(n_{III}|s_yc_y\omega),$$

$$H(n_{III}|s_xs_xs_x)=-\frac{1}{8k_x^2}G(n_{III}|s_xs_xs_x),\qquad H(n_{III}|s_xs_xc_x)=-\frac{1}{8k_x^2}G(n_{III}|s_xs_xc_x),$$

$$H(n_{III}|s_xs_ys_y)=\frac{1}{4q_5}G(n_{III}|s_xs_ys_y)+\frac{k_x}{4k_yq_5}G(n_{III}|c_xs_yc_y),\qquad H(n_{III}|s_xs_yc_y)=\frac{1}{4q_5}G(n_{III}|s_xs_yc_y)-\frac{k_x}{4k_yq_5}((n_{III}|c_xs_ys_y),$$

$$H(n_{III}|c_xs_ys_y)=\frac{1}{4q_5}G(n_{III}|c_xs_ys_y)-\frac{k_x}{4k_yq_5}G(n_{III}|s_xs_yc_y),\qquad H(n_{III}|c_xs_yc_y)=\frac{1}{4q_5}G(n_{III}|c_xs_yc_y)+\frac{k_x}{4k_yq_5}G(n_{III}|s_xs_ys_y)\;.$$

(5-34)

The eq.(5-33) can be rearranged as

$$x_{III}(\omega) = \sum_{n_{III}} n_{III} \sum_{m} m\cdot H(n_{III}|m)$$

$$= \sum_{n_{III}} n_{III}\cdot R(x|n_{III}) \qquad (5\text{-}35)$$

In this expression the nineteen $R(x|n_{III})$ are directly the third order elements (nos. 13 to 31) of the first row of the radial transfer matrix as shown in fig.5,p42. The n_{III} as given in eq.(5-31) characterizes the appropriate column.

The third order term of tan(α), $(\tan\alpha)_{III}$ is obtained from the relationship eq.(5-12a) as

$$(\tan\alpha)_{III} = \frac{1}{\rho_0{}^2}\left[-x_I x'_{II} - x'_I x_{II} + \frac{1}{\rho_0} x_I^2 x'_I \right] + \frac{1}{\rho_0} x'_{III} \qquad (5\text{-}36)$$

Substituting eqs.(5-17a, 5-26a, 5-35) in this expression and rearranging this we obtain

$$(\tan\alpha)_{III} = \sum_{n_{III}} n_{III} \sum_{m} m\, H'(n_{III}|m)$$

$$= \sum_{n_{III}} n_{III} \sum_{m} m \left[H'_1(n_{III}|m) + H'_2(n_{III}|m) \right]$$

$$= \sum_{n_{III}} n_{III}\cdot R(\alpha|n_{III}) \;. \qquad (5\text{-}37)$$

Here H'_1 is derived from $\frac{1}{\rho_0{}^2}[-x_I x'_{II} - x'_I x_{II} + \frac{1}{\rho_0} x_I^2 x'_I]$ and H'_2 from $\frac{1}{\rho_0} x'_{III}$. Both H' comprise the elements of a 19 x 22 matrix.

	1	ω	s_x	c_x	$s_x\omega$	$c_x\omega$	s_xs_x	s_xc_x	s_ys_y	s_yc_y	$s_x\omega\omega$	$c_x\omega\omega$	$s_xs_x\omega$	$s_xc_x\omega$	$s_ys_y\omega$	$s_yc_y\omega$	$s_xs_xs_x$	$s_xs_xc_x$	$s_xs_ys_y$	$s_xs_yc_y$	$c_xs_ys_y$	$c_xs_yc_y$
xxx			H'			H'		H'									H'					
xxα	H'			H'	H'		H'	H'										H'				
xxδ		H'	H'			H'		H'					H'				H'					
xαα	H'		H'	H'		H'	H'	H'									H'					
xαδ	H'			H'	H'		H'							H'				H'				
xδδ		H'	H'			H'		H'			H'		H'				H'					
xyy			H'			H'		H'		H'									H'			H'
xyβ	H'			H'	H'		H'		H'											H'	H'	
xββ	H'		H'			H'	H'	H'		H'									H'			H'
ααα	H'			H'	H'		H'											H'				
ααδ	H'	H'	H'			H'	H'	H'					H'				H'					
αδδ	H'		H'	H'	H'		H'				H'	H'	H'	H'				H'				
αyy	H'			H'	H'		H'		H'											H'	H'	
αyβ			H'			H'		H'		H'									H'			H'
αββ	H'			H'	H'		H'		H'											H'	H'	
δδδ		H'	H'	H'	H'	H'		H'			H'		H'				H'					
δyy	H'	H'	H'	H'		H'	H'	H'		H'					H'				H'			H'
δyβ	H'			H'	H'		H'		H'							H'				H'	H'	
δββ		H'	H'			H'		H'		H'					H'				H'			H'

As above all non-zero elements are indicated by H'. Explicitly the $H'(n_{III}|m)$ read :

$$H_1'(xxx|s_x) = \frac{1}{3\rho_0^3}\left(2k_x - \frac{7q_1}{k_x}\right), \qquad H_1'(xxx|s_xc_x) = \frac{2}{3\rho_0^3}\left(-k_x + \frac{5q_1}{k_x}\right), \qquad H_1'(xxx|s_xs_xs_x) = \frac{1}{\rho_0^3}\left(-k_x + \frac{q_1}{k_x}\right),$$

$$H_1'(xx\alpha|1) = \frac{\rho_0}{k_x} H_1'(xxx|s_x), \qquad H_1'(xx\alpha|c_x) = -H_1'(xx\alpha|1), \qquad H_1'(xx\alpha|s_xs_x) = -2H_1'(xx\alpha|1),$$

$$H_1'(xx\alpha|s_xs_xc_x) = -\frac{3\rho_0}{k_x} H_1'(xxx|s_xs_xs_x), \qquad H_1'(xx\delta|\omega) = -\frac{1}{2\rho_0^2}\left(k_x^2 + 1 + \frac{2q_1}{k_x^2}\right),$$

$$H_1'(xx\delta|s_x) = \frac{1}{3k_x\rho_0^2}\left(-7 + \frac{26q_1}{k_x^2}\right), \qquad H_1'(xx\delta|s_xc_x) = \frac{1}{\rho_0^2}\left(-\frac{k_x}{2} + \frac{5}{6k_x} - \frac{35q_1}{3k_x^2}\right),$$

$$H_1'(xx\delta|s_xs_x\omega) = -2H_1'(xx\delta|\omega), \qquad H_1'(xx\delta|s_xs_xs_x) = \frac{1}{k_x} H_1'(xx\alpha|s_xs_xc_x), \qquad H_1'(x\alpha\alpha|s_x) = \frac{1}{3k_x\rho_0}\left(-7 + \frac{8q_1}{k_x^2}\right),$$

$$H_1'(x\alpha\alpha|s_xc_x) = \frac{4}{3k_x\rho_0}\left(1 - \frac{2q_1}{k_x^2}\right), \qquad H_1'(x\alpha\alpha|s_xs_xs_x) = \rho_0 H_1'(xx\delta|s_xs_xs_x), \qquad H_1'(x\alpha\delta|1) = \frac{1}{\rho_0}\left(-\frac{5}{3k_x^2} + \frac{16q_1}{3k_x^4}\right),$$

$$H_1'(x\alpha\delta|c_x) = -H_1'(x\alpha\delta|1), \qquad H_1'(x\alpha\delta|s_xs_x) = -2H_1'(x\alpha\delta|1), \qquad H_1'(x\alpha\delta|s_xc_x\omega) = \frac{4\rho_0}{k_x} H_1'(xx\delta|\omega),$$

$$H_1'(x\alpha\delta|s_xs_xc_x) = \frac{6\rho_0^2}{k_x^3} H_1'(xxx|s_xs_xs_x), \qquad H_1'(x\delta\delta|\omega) = -\frac{2\rho_0}{k_x^2} H_1'(xx\delta|\omega),$$

$$H_1'(x\delta\delta|s_x) = \frac{1}{\rho_0}\left(-\frac{1}{2k_x} + \frac{19}{6k_x^3} - \frac{31q_1}{3k_x^5}\right), \qquad H_1'(x\delta\delta|c_x\omega) = \frac{\rho_0}{k_x^2} H_1'(xx\delta|\omega),$$

$$H_1'(x\delta\delta|s_xc_x) = \frac{1}{\rho_0}\left(\frac{1}{k_x} - \frac{11}{3k_x^3} + \frac{34q_1}{3k_x^5}\right), \qquad H_1'(x\delta\delta|s_xs_x\omega) = -2H_1'(x\delta\delta|\omega),$$

$$H_1'(x\delta\delta|s_xs_xs_x) = \frac{3\rho_0^2}{k_x^4} H_1(xxx|s_xs_xs_x), \qquad H_1'(xyy|s_x) = \frac{1}{\rho_0^3}\left(-\frac{k_y^2}{k_x} + \frac{q_4q_6}{k_xq_7}\right), \qquad H_1'(xyy|s_xc_x) = -2H_1'(xyy|s_x),$$

$$H_1'(xyy|s_xs_ys_y) = -\frac{k_xq_4}{\rho_0^3q_7}, \qquad H_1'(xyy|c_xs_yc_y) = -\frac{2k_y}{k_x} H_1'(xyy|s_xs_ys_y), \qquad H_1'(xy\beta|1) = \frac{2q_4}{\rho_0^2q_7},$$

$$H_1'(xy\beta|c_x) = -H_1'(xy\beta|1), \qquad H_1'(xy\beta|s_xs_x) = -2H_1'(xy\beta|1), \qquad H_1'(xy\beta|s_xs_yc_y) = \frac{k_x}{k_y} H_1'(xy\beta|1),$$

$$H_1'(xy\beta|c_xs_ys_y) = 2H_1'(xy\beta|1), \qquad H_1'(x\beta\beta|s_x) = \frac{1}{\rho_0}\left(-\frac{1}{k_x} - \frac{2q_4}{k_xq_7}\right), \qquad H_1'(x\beta\beta|s_xc_x) = -2H_1'(x\beta\beta|s_x),$$

$$H_1'(x\beta\beta|s_xs_ys_y) = \frac{\rho_0}{2k_y} H_1'(xy\beta|s_xs_yc_y), \qquad H_1'(x\beta\beta|c_xs_yc_y) = -\frac{\rho_0}{k_x} H_1'(xy\beta|s_xs_yc_y),$$

$$H_1'(\alpha\alpha\alpha|1) = -\frac{\rho_0}{4k_x} H_1'(x\alpha\alpha|s_xc_x), \quad H_1'(\alpha\alpha\alpha|c_x) = -H_1'(\alpha\alpha\alpha|1), \quad H_1'(\alpha\alpha\alpha|s_xs_x) = -2H_1'(\alpha\alpha\alpha|1),$$

$$H_1'(\alpha\alpha\alpha|s_xs_xc_x) = \frac{\rho_0^3}{k_x^3} H_1'(xxx|s_xs_xs_x), \quad H_1'(\alpha\alpha\delta|\omega) = -\frac{\rho_0^2}{k_x^2} H_1'(xx\delta|\omega), \quad H_1'(\alpha\alpha\delta|s_x) = \frac{4}{k_x} H_1'(\alpha\alpha\alpha|s_xs_x),$$

$$H_1'(\alpha\alpha\delta|s_xc_x) = \left(-\frac{1}{2k_x} - \frac{19}{6k_x^3} + \frac{7q_1}{3k_x^5}\right), \quad H_1'(\alpha\alpha\delta|s_xs_x\omega) = -2H_1'(\alpha\alpha\delta|\omega),$$

$$H_1'(\alpha\alpha\delta|s_xs_xs_x) = \frac{3}{k_x} H_1'(\alpha\alpha\alpha|s_xs_xc_x), \quad H_1'(\alpha\delta\delta|1) = \left(\frac{3}{k_x^4} - \frac{2q_1}{k_x^6}\right), \quad H_1'(\alpha\delta\delta|c_x) = -H_1'(\alpha\delta\delta|1),$$

$$H_1'(\alpha\delta\delta|s_xs_x) = -2H_1'(\alpha\delta\delta|1), \quad H_1'(\alpha\delta\delta|s_x\omega) = -\frac{1}{k_x} H_1'(\alpha\alpha\delta|\omega), \quad H_1(\alpha\delta\delta|s_xc_x\omega) = -4H_1'(\alpha\delta\delta|s_x\omega),$$

$$H_1'(\alpha\delta\delta|s_xs_xc_x) = -\frac{1}{k_x} H_1'(\alpha\alpha\delta|s_xs_xs_x), \quad H_1'(\alpha yy|1) = \frac{\rho_0}{k_x} H_1'(xyy|s_x), \quad H_1'(\alpha yy|c_x) = -H_1'(\alpha yy|1),$$

$$H_1'(\alpha yy|s_xs_x) = -2H_1'(\alpha yy|1), \quad H_1'(\alpha yy|s_xs_yc_y) = \frac{\rho_0}{k_x} H_1'(xyy|c_xs_yc_y), \quad H_1'(\alpha yy|c_xs_ys_y) = \tfrac{1}{2}H_1'(xy\beta|1),$$

$$H_1'(\alpha y\beta|s_x) = -\frac{\rho_0}{k_y} H_1'(\alpha yy|s_xs_yc_y), \quad H_1'(\alpha y\beta|s_xc_x) = -2H_1'(\alpha y\beta|s_x), \quad H_1'(\alpha y\beta|s_xs_ys_y) = H_1'(\alpha y\beta|s_xc_x),$$

$$H_1'(\alpha y\beta|c_xs_yc_y) = -\frac{2\rho_0}{k_y} H_1'(\alpha yy|c_xs_ys_y), \quad H_1'(\alpha\beta\beta|1) = \frac{\rho_0}{k_x} H_1'(x\beta\beta|s_x), \quad H_1'(\alpha\beta\beta|c_x) = -H_1'(\alpha\beta\beta|1),$$

$$H_1'(\alpha\beta\beta|s_xs_x) = \left(\frac{2}{k_x^2} + \frac{4q_1}{k_x^4q_7}\right), \quad H_1'(\alpha\beta\beta|s_xs_yc_y) = \frac{\rho_0}{k_x} H_1'(x\beta\beta|c_xs_yc_y), \quad H_1'(\alpha\beta\beta|c_xs_ys_y) = -\frac{\rho_0}{k_y^2} H_1'(\alpha yy|c_xs_ys_y),$$

$$H_1'(\delta\delta\delta|\omega) = \frac{1}{k_x} H_1'(\alpha\delta\delta|s_x\omega), \quad H_1'(\delta\delta\delta|s_x) = \left(\frac{1}{2k_x^3} - \frac{7}{2k_x^5} + \frac{3q_1}{k_x^7}\right), \quad H_1'(\delta\delta\delta|c_x\omega) = -H_1'(\delta\delta\delta|\omega),$$

$$H_1'(\delta\delta\delta|s_xc_x) = -H_1'(\delta\delta\delta|s_x), \quad H_1'(\delta\delta\delta|s_xs_x\omega) = 2H_1'(\delta\delta\delta|c_x\omega), \quad H_1'(\delta\delta\delta|s_xs_xs_x) = \frac{1}{3k_x} H_1'(\alpha\delta\delta|s_xs_xc_x),$$

$$H_1'(\delta yy|s_x) = \frac{1}{k_x} H_1'(\alpha yy|s_xs_x), \quad H_1'(\delta yy|s_xc_x) = -H_1'(\delta yy|s_x), \quad H_1'(\delta yy|s_yc_y) = \frac{1}{k_x} H_1'(\alpha yy|s_xs_yc_y),$$

$$H_1'(\delta yy|s_xs_ys_y) = \frac{1}{k_x} H_1'(\alpha yy|c_xs_ys_y), \quad H_1'(\delta yy|c_xs_yc_y) = -H_1'(\delta yy|s_yc_y), \quad H_1'(\delta y\beta|1) = -\frac{4\rho_0}{k_x} H_1'(\delta yy|s_xs_ys_y),$$

$$H_1'(\delta y\beta|c_x) = -H_1'(\delta y\beta|1), \quad H_1'(\delta y\beta|s_xs_x) = H_1'(\delta y\beta|c_x), \quad H_1'(\delta y\beta|s_ys_y) = H_1'(\delta y\beta|c_x),$$

$$H_1'(\delta y\beta|s_xs_yc_y) = \frac{1}{\rho_0} H_1'(\alpha\beta\beta|s_xs_yc_y), \quad H_1'(\delta y\beta|c_xs_ys_y) = -H_1'(\delta y\beta|s_xs_x), \quad H_1'(\delta\beta\beta|s_x) = \frac{\rho_0}{k_x^2} H_1'(x\beta\beta|s_xc_x),$$

$$H_1'(\delta\beta\beta|s_xc_x) = -H_1'(\delta\beta\beta|s_x), \quad H_1'(\delta\beta\beta|s_yc_y) = \frac{\rho_0}{k_x} H_1'(\delta y\beta|s_xs_yc_y), \quad H_1'(\delta\beta\beta|s_xs_ys_y) = \frac{1}{2k_y} H_1'(\alpha\beta\beta|s_xs_yc_y),$$

$$H_1'(\delta\beta\beta|c_xs_yc_y) = -H_1'(\delta\beta\beta|s_yc_y);$$

$$H_2'(n_{\text{III}}|1) = [H(n_{\text{III}}|\omega) + k_xH(n_{\text{III}}|s_xc_x) + k_yH(n_{\text{III}}|s_yc_y)]/\rho_0, \quad H_2'(n_{\text{III}}|\omega) = [k_xH(n_{\text{III}}|s_xc_x\omega) + k_yH(n_{\text{III}}|s_yc_y\omega)]/\rho_0,$$

$$H_2'(n_{\text{III}}|s_x) = [-k_xH(n_{\text{III}}|c_x) + H(n_{\text{III}}|s_x\omega) + 2k_xH(n_{\text{III}}|s_xs_xc_x) + k_yH(n_{\text{III}}|s_xs_yc_y)]/\rho_0,$$

$$H_2'(n_{\text{III}}|c_x) = [k_xH(n_{\text{III}}|s_x) + H(n_{\text{III}}|c_x\omega) + k_yH(n_{\text{III}}|c_xs_yc_y)]/\rho_0,$$

$$H_2'(n_{\text{III}}|s_x\omega) = [-k_xH(n_{\text{III}}|c_x\omega) + 2H(n_{\text{III}}|s_x\omega\omega)]/\rho_0, \quad H_2'(n_{\text{III}}|c_x\omega) = [k_xH(n_{\text{III}}|s_x\omega) + 2H(n_{\text{III}}|c_x\omega\omega)]/\rho_0,$$

$$H_2'(n_{\text{III}}|s_xs_x) = [-2k_xH(n_{\text{III}}|s_xc_x) + H(n_{\text{III}}|s_xs_x\omega)]/\rho_0, \quad H_2'(n_{\text{III}}|s_xc_x) = [2k_xH(n_{\text{III}}|s_xs_x) + H(n_{\text{III}}|s_xc_x\omega)]/\rho_0,$$

$$H_2'(n_{\text{III}}|s_ys_y) = [-2k_yH(n_{\text{III}}|s_yc_y) + H(n_{\text{III}}|s_ys_y\omega)]/\rho_0, \quad H_2'(n_{\text{III}}|s_yc_y) = [2k_yH(n_{\text{III}}|s_ys_y) + H(n_{\text{III}}|s_yc_y\omega)]/\rho_0,$$

$$H_2'(n_{\text{III}}|s_x\omega\omega) = -k_xH(n_{\text{III}}|c_x\omega\omega), \quad H_2'(n_{\text{III}}|c_x\omega\omega) = k_xH(n_{\text{III}}|s_x\omega\omega)/\rho_0,$$

$$H_2'(n_{\text{III}}|s_xs_x\omega) = -2k_xH(n_{\text{III}}|s_xc_x\omega)/\rho_0, \quad H_2'(n_{\text{III}}|s_xc_x\omega) = 2k_xH(n_{\text{III}}|s_xs_x\omega)/\rho_0,$$

$$H_2'(n_{\text{III}}|s_ys_y\omega) = -2k_yH(n_{\text{III}}|s_yc_y\omega)/\rho_0, \quad H_2'(n_{\text{III}}|s_yc_y\omega) = 2k_yH(n_{\text{III}}|s_ys_y\omega)/\rho_0,$$

$$H_2'(n_{\text{III}}|s_xs_xs_x) = -3k_xH(n_{\text{III}}|s_xs_xc_x)/\rho_0, \quad H_2'(n_{\text{III}}|s_xs_xc_x) = 3k_xH(n_{\text{III}}|s_xs_xs_x)/\rho_0,$$

$$H_2'(n_{\text{III}}|s_xs_ys_y) = [-2k_yH(n_{\text{III}}|s_xs_yc_y) - k_xH(n_{\text{III}}|c_xs_ys_y)]/\rho_0,$$

$$H_2'(n_{\text{III}}|s_xs_yc_y) = [2k_yH(n_{\text{III}}|s_xs_ys_y) - k_xH(n_{\text{III}}|c_xs_yc_y)]/\rho_0,$$

$$H_2'(n_{\text{III}}|c_xs_ys_y) = [k_xH(n_{\text{III}}|s_xs_ys_y) - 2k_yH(n_{\text{III}}|c_xs_yc_y)]/\rho_0,$$

$$H_2'(n_{\text{III}}|c_xs_yc_y) = [k_xH(n_{\text{III}}|s_xs_yc_y) + 2k_yH(n_{\text{III}}|c_xs_ys_y)]/\rho_0 \qquad (5\text{-}38)$$

The nineteen $R(\alpha|n_{III})$ in eq.(5-37) represent the third order elements (nos. 13 to 31) of the second row of the radial transfer matrix as shown in fig.5. Here again n_{III} in eq.(5-31) characterizes the appropriate column.

5.6. Transfer matrices for a toroidal condenser

The results of sections 5.3, 5.4 and 5.5 allow us to determine the coordinates x,y and the corresponding angles of inclination α,β in a toroidal condenser with high precision if we know the initial values of $x=x_2$, $y=y_2$, $\alpha=\alpha_2$, $\beta=\beta_2$ at the entrance of the toroidal condenser. It also is necessary to know the relative energy deviation $\delta=(U-U_0)/U_0$ of the particle under consideration. Here it is assumed that from this initial position the particles will fly in the potential distribution of a 360° toroidal condenser.

In detail we can describe the $x(\omega)$, $\alpha(\omega)$ to a third order approximation

$$^3x(\omega) = x_I + x_{II} + x_{III} = \sum_{t_x} t_x R(x|t_x) , \qquad (5\text{-}39a)$$

$$^3\tan\alpha(\omega) = (\tan\alpha)_I + (\tan\alpha)_{II} + (\tan\alpha)_{III} = \sum_{t_x} t_x R(\alpha|t_x) , \qquad (5\text{-}40a)$$

where

t_x=x, tanα, δ, xx, xα, xδ, αα, αδ, δδ, yy, yβ, ββ, xxx, xxα, xxδ, xαα, xαδ, xδδ, xyy, xyβ, xββ, ααα, ααδ, αδδ, αyy, αyβ, αββ, δδδ, δyy, δyβ, δββ, (5-41a)

and where the $R(x|t_x)$ and $R(\alpha|t_x)$ are given by eqs.(5-17a), (5-18a), (5-26a) (5-27a), (5-35) and (5-37). The $y(\omega)$, $\beta(\omega)$ can be described to a second order approximation.

$$^2y(\omega) = y_I + y_{II} = \sum_{t_y} t_y R(y|t_y) , \qquad (5\text{-}39b)$$

$$^2\beta(\omega) = \beta_I + \beta_{II} = \sum_{t_y} t_y R(\beta|t_y) , \qquad (5\text{-}40b)$$

where

t_y = y, β, yx, yα, yδ, βx, βα, βδ, (5-41b).

and where the $R(y|t_y)$ and $R(\beta|t_y)$ are given by eqs.(5-17b), (5-18b), (5-26b) and (5-27b).

If the x, α, y, β, δ of eqs.(5-41) are taken at an inital position ω =0 the eqs.(5-39) yield $x(\omega)$, $\alpha(\omega)$, $y(\omega)$ and $\beta(\omega)$ at a position ω since the $R(x|t)$, $R(y|t)$ depend on ω. t_x and t_y are called the radial and axial position vectors. The $R(i|t)$ then can be understood as elements of so called transfer matrices which transform the vectors t_x and t_y from a position $\omega=0$ to a position ω.

The radial transfer matrix ${}^3T$ contains 31 x 31 elements ${}^3R(t_x|t_x)$. The axial transfer matrix ${}^2T_y$ contains 8 x 8 elements ${}^2R(t_y|t_y)$. Since only the first two elements of the vectors t_x or t_y are independent variables it is evident that the matrix elements of the rows of higher index than 3 are functions of the elements of the first 2 rows.

The axial transfer matrix ${}^2T_y$ is shown in fig.4.p31. The elements of the first two rows are given by eqs.(5-17b), (5-18b), (5-26b) and (5-27b). The other are defined as

$$R(ij|mn) = R(i|m)\ R(j|n) \tag{5-42}$$

where $i,m = y,\beta$, $\quad j,n = x,\alpha,\delta$.
The radial transfer matrix ${}^3T_x$ is shown in fig.5.p42. The elements of the first two rows are given by eqs.(5-17a), (5-18a), (5-26a), (5-27a), (5-35) and (5-37). The other

For the rows 4 to 12 we have

$$R(ij|mn) = \sum R(i|m)R(j|n) \tag{5-43}$$

where the summation symbol means the summation for all possible combinations of m, n which yield the same product mn.

For the rows 4 to 9 (i.e. rows xx, xα, xδ, αα, αδ, δδ) the i, j and m, n stand for

$$i,j = x,\ \alpha,\ \delta, \qquad m,n = x,\ \alpha,\ \delta,\ xx,\ x\alpha,\ x\delta,\ \alpha\alpha,\ \alpha\delta,\ \delta\delta,\ yy,\ y\beta,\ \beta\beta$$

for the rows 10 to 12 (i.e. rows yy, yβ, ββ) the i,j, and m,n stand for

$$i,j = y,\beta, \qquad m,n = y,\ \beta,\ yx,\ y\alpha,\ y\delta,\ \beta x,\ \beta\alpha,\ \beta\delta.$$

The corresponding $R(i|m)$ and $R(j|m)$ are to be taken from eqs.(5-17b), (5-18b), (5-26b) and (5-27b).

Analogously we can determine the $R(s|t)$ for the rows 13 to 31 (i.e. rows xxx, xxα, xxδ, xαα, xαδ, xδδ, xyy, xyβ, yββ, ααα, ααδ, αyy, αyβ, αββ, δδδ, δyy, δyβ, δββ) as

$$R(ijk|lmn) = \sum R(i|l)R(j|m)R(k|n) \tag{5-44}$$

and the summation should be done for all possible combinations of l, m, n among group x, α, δ or y, β.
The i, j, k, l, m, n stand for

$$i,l = x,\alpha,\delta, \qquad j,k,m,n = x,\alpha,\delta,y,\beta.$$

The product of the third order fringing field matrices for an electrostatic condenser and the matrices T_y and T_x of figs.4 and 5 describes the radial motion of charged particles through a real electrostatic sector field to third order.

	x	α	δ	xx	xα	xδ	αα	αδ	δδ	yy	yβ	ββ	xxx	xxα	xxδ	xαα	xαδ	xδδ	xyy	xyβ	xββ	ααα	ααδ	αδδ	αyy	αyβ	αββ	δδδ	δyy	δyβ	δββ
x	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
α	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
δ			1																												
xx				R	R	R	R	R	R				R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
xα				R	R	R	R	R	R				R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
xδ						R		R	R						R		R	R					R	R				R	R	R	R
αα				R	R	R	R	R	R				R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
αδ						R		R	R						R		R	R					R	R				R	R	R	R
δδ									1																						
yy										R	R	R							R	R	R				R	R	R		R	R	R
yβ										R	R	R							R	R	R				R	R	R		R	R	R
ββ										R	R	R							R	R	R				R	R	R		R	R	R
xxx													R	R	R	R	R	R				R	R	R				R			
xxα													R	R	R	R	R	R				R	R	R				R			
xxδ															R		R	R					R	R				R			
xαα													R	R	R	R	R	R				R	R	R				R			
xαδ															R		R	R					R	R				R			
xδδ																		R						R				R			
xyy																			R	R	R				R	R	R		R	R	R
xyβ																			R	R	R				R	R	R		R	R	R
xββ																			R	R	R				R	R	R		R	R	R
ααα													R	R	R	R	R	R				R	R	R				R			
ααδ															R		R	R					R	R				R			
αδδ																		R						R				R			
αyy																			R	R	R				R	R	R		R	R	R
αyβ																			R	R	R				R	R	R		R	R	R
αββ																			R	R	R				R	R	R		R	R	R
δδδ																												1			
δyy																													R	R	R
δyβ																													R	R	R
δββ																													R	R	R

Fig.5. The radial transfer matrix of third order. The elements of the first and the second row are given by eqs.(5-17a), (5-18a), (5-26a), (5-35) and (5-37). The other matrix elements are defined by eqs.(5-43).

[6] Particle Trajectories in an Inhomogeneous Sector Magnet

This chapter deals with the ion trajectory in an inhomogeneous magnetic field. The use of such field for a mass spectrometer may result in a greatly enlarged mass dispersion and resolving power in comparison with the ordinary type. Here the path equation is derived by using Lagrange's equations. There are many common equations in both electric and magnetic field. Such parts are omitted in this chapter and the detailed discussions are given in ref.9).

6.1. The Field Distribution in an Inhomogeneous Magnetic Field

The simplest way to make inhomogeneous fields is the use of conical pole faces, whose radial section is shown in fig.6.p47. In order to calculate the ion trajectory in a third order approximation, the magnetic field in the conical pole faces should be known as the power series expression of fourth order. The method to obtain expansion coefficients has already been established in a second order approximation.[10)] We will here introduce the main results and make some extension to the fourth order calculation. We assume that the magnetic field in the plane z=0 (median plane) is expressed by the following Taylor expansion :

$$B_y(x,0) = B_0\left[1 - n_1\left(\frac{x}{\rho_0}\right) - n_2\left(\frac{x}{\rho_0}\right)^2 - n_3\left(\frac{x}{\rho_0}\right)^3 + \cdots\cdots \right] \qquad (6\text{-}1a)$$

$$B_x(x,0) = 0 \qquad (6\text{-}1b)$$

where B_0 is the field strength along the circular circumference x=0 in the median plane and the coefficients n_1, n_2 and n_3 are determined as the function of geometrical parameters. They may be expanded in a power series of half gap distance $\rho_0 d$ at x=0.

From symmetry we have

$$n_k = n_{k0} + n_{k2}d^2 + n_{k4}d^4 + \cdots\cdots \qquad (k=1,2,3, \cdots) \qquad (6\text{-}2)$$

A conical pole face may be represented by :

$$\frac{y}{\rho_0} = \left(p\frac{x}{\rho_0} + 1\right)d \quad , \qquad (6\text{-}3)$$

where p and d are constants. Eq.(6-3) should give an equipotential surface for the scalar magnetic potential Φ_m. Since the scalar magnetic potential on the conical pole face should be independent of x, we can determine the coefficients n_{ki} as the function of p. After some tedious calculation we have the following relationships :

$$n_{10} = p\ ,$$

$$n_{12} = -\frac{p(1-p)}{6}\ ,$$

$$n_{14} = \frac{p(1-p)}{360}\,[-21 + 2p + 36p^2]$$

$$n_{20} = -p^2$$

$$n_{22} = \frac{p(1-p^2)}{6}$$

$$n_{24} = -\frac{p(1-p)}{360}\,[-42 + 8p + 12p^2 + 36p^3]$$

$$n_{30} = p^3$$

$$n_{32} = -\frac{p(1-p^3)}{6}$$

$$n_{34} = \frac{p(1-p)}{360}\,[-70 + 28p - 9p^2 + 12p^3 + 36p^4]$$

(6-4)

Introducing eqs.(6-4) into eq.(6-2), we obtain the coefficients n_1, n_2, n_3. The coefficient n_1 is related to the so-called inhomogeneity factor or field index n which is defined as $n=\frac{dB/B}{dr/r}$ by the relationship $n=-n_1$.

Then, using Maxwellian equations div $\mathbf{B}$ = rot $\mathbf{B}$ = 0 , the third order expression of the magnetic field is obtained as :

$$B_x(x,y) = B_0[\, n_1(\frac{y}{\rho_0}) + 2n_1(\frac{x}{\rho_0})(\frac{y}{\rho_0}) + 3n_2(\frac{x}{\rho_0})^2(\frac{y}{\rho_0}) + \frac{1}{6}(n_1-2n_2-6n_3)(\frac{y}{\rho_0})^3 + \cdots\cdots] \qquad (6\text{-}5a)$$

$$B_y(x,y) = B_0\left[1 + n_1\left(\frac{x}{\rho_0}\right) - n_2\left(\frac{x}{\rho_0}\right)^2 - \frac{1}{2}(n_1+2n_2)\left(\frac{y}{\rho_0}\right)^2 + n_3\left(\frac{x}{\rho_0}\right)^3 + \frac{1}{2}(n_1-2n_2-6n_3)\left(\frac{x}{\rho_0}\right)\left(\frac{y}{\rho_0}\right)^2 + \cdots\cdots \right]. \qquad (6\text{-}5b)$$

The vector potential which yields the above field has only ω-component A_ω and is given by

$$A_\omega(x,y) = \frac{1}{2}\rho_0 B_0\left[1 + \left(\frac{x}{\rho_0}\right) + n_1\left(\frac{x}{\rho_0}\right)^2 - n_1\left(\frac{y}{\rho_0}\right)^2 + \frac{1}{3}(2n_2-n_1)\left(\frac{x}{\rho_0}\right)^3 - 2n_2\left(\frac{x}{\rho_0}\right)\left(\frac{y}{\rho_0}\right)^2 + \frac{1}{6}(2n_1-n_2+3n_3)\left(\frac{x}{\rho_0}\right)^4 - 3n_3\left(\frac{x}{\rho_0}\right)^2\left(\frac{y}{\rho_0}\right)^2 - \frac{1}{12}(n_1-2n_2-6n_3)\left(\frac{y}{\rho_0}\right)^4 + \cdots\cdots \right] \qquad (6\text{-}6)$$

6.2. The Third Order Path Equation

The Lagrangian L is given by :

$$L = \frac{1}{2}m[\dot{x}^2 + (\rho_0+x)^2\dot{\omega}^2 + \dot{y}^2] + e(\rho_0+x)\dot{\omega}A_\omega \qquad (6\text{-}7)$$

where m and e are the mass and the charge of ion respectively. Then the Lagrange equations now become

$$m\ddot{x} = m(\rho_0+x)\omega^2 + e(\rho_0+x)\dot{\omega}B_y \qquad (6\text{-}8a)$$

$$\frac{d}{dt}[m(\rho_0+x)^2\dot{\omega} + e(\rho_0+x)A_\omega] = 0 \qquad (6\text{-}8b)$$

$$m\ddot{y} = -e(\rho_0+x)\dot{\omega}B_x. \qquad (6\text{-}8c)$$

We assume that the ion moving along the central path has the mass m_0, momentum p_0 and charge e, and we want to study the motion of an ion having the mass $m=m_0(1+\gamma)$, momentum $p=p_0(1+\tau)$ and charge e. Because eq.(6-8b) can be integrated easily, $\dot{\omega}$ can be determined under suitable initial condition. If we substitute this $\dot{\omega}$ and eq.(6-5b) for B_y into eq.(6-8a), we obtain the equation of motion in the radial direction. However, since the path of the ion is what we wish to ascertain, we must replace d/dt by d/dω by using the relation given in eqs.(2-31). Thus the third order path equation in the radial direction is obtained as follows :

$$\frac{d^2x}{d\omega^2} + k_x{}^2 x = \tau\rho_0 - \frac{1}{2}(1+n_1)x_2{}^2/\rho_0 + \frac{1}{2}n_1 y_2{}^2/\rho_0 - \frac{1}{2}(\alpha_2{}^2+\beta_2{}^2) + x_2\tau - \tau^2\rho_0$$
$$- \frac{1}{3}(n_1+n_2)x_2{}^3/\rho_0{}^2 - \frac{1}{2}(x_2-\tau\rho_0)(\alpha_2{}^2+\beta_2{}^2) + \frac{1}{2}(n_1+2n_2)x_2 y_2{}^2/\rho_0{}^2$$
$$+ (1+n_1)x_2{}^2\tau/\rho_0 - n_1 y_2{}^2\tau/\rho_0 - 2x_2\tau^2 + \tau^3\rho_0 + q_1 x^2/\rho_0 + n_2 y^2/\rho_0$$
$$+ (1+n_1)x\tau + x[-\frac{1}{2}(1+n_1)^2 x_2{}^2/\rho_0{}^2 + \frac{1}{2}n_1(1+n_1)y_2{}^2/\rho_0{}^2$$
$$- \frac{1}{2}(1+n_1)(\alpha_2{}^2+\beta_2{}^2) + (1+n_1)x_2\tau/\rho_0 - n_1\tau^2] + x^2/\rho_0 \; [-(1-n_2)\tau]$$
$$+ y^2/\rho_0[(\frac{1}{2}n_1-n_2)\tau] + q_2 x^3/\rho_0{}^2 + q_3 xy^2/\rho_0{}^2 + (\frac{dx}{d\omega})^2/\rho_0{}^2 \; [1$$
$$-(n_1+2)x/\rho_0+\tau] + n_1 y(dx/d\omega)(dy/d\omega)/\rho_0{}^2 \quad . \qquad (6\text{-}9a)$$

The second order path equation in the axial direction is obtained similarly from eq.(6-8c) and is given by

$$\frac{d^2y}{d\omega^2} + k_y{}^2 y = 2(n_1+n_2)xy/\rho_0 - n_1 y\tau + (\frac{dx}{d\omega})(\frac{dy}{d\omega})/\rho_0 \quad , \qquad (6\text{-}9b)$$

where

$$k_x{}^2 = 1 + n_1$$

$$k_y{}^2 = - n_1$$

$$q_1 = - \frac{1}{2}(1 + 3n_1 + 2n_2)$$

$$q_2 = \frac{1}{2} + \frac{1}{3} n_1 + \frac{1}{2} n_1{}^2 - \frac{5}{3} n_2 - n_3$$

$$q_3 = - \frac{1}{2} n_1(1+n_1) + 2n_2 + 3n_3 \quad . \qquad (6\text{-}10)$$

6.3. The Solution of the Path Equation

6.3.1. Radial Direction

In the case we can also assume that the path equations are expressed in the following form : we conbine this time the first, the second and the third order terms into one equation.

$$\frac{d^2x}{d\omega^2} + k_x{}^2x = \sum_m m \sum_n n\ G(n|m) \tag{6-11}$$

where the meaning of the vector m, n and the matrix $G(n|m)$ are the same as defined in chapter 5.
Writing again,

$$\begin{aligned} m = {} & 1,\ \omega,\ s_x,\ c_x,\ s_x\omega,\ c_x\omega,\ s_x{}^2,\ s_xc_x,\ s_y^2,\ s_yc_y, \\ & s_x\omega^2,\ c_x\omega^2,\ s_x{}^2\omega,\ s_xc_x\omega,\ s_y{}^2\omega,\ s_yc_y\omega,\ s_x{}^3,\ s_x{}^2c_x, \\ & s_xs_y{}^2,\ s_xs_yc_y,\ c_xs_y{}^2,\ c_xs_yc_y, \end{aligned} \tag{6-12}$$

$$\begin{aligned} n = {} & x,\ \alpha,\ \tau,\ xx,\ x\alpha,\ x\tau,\ \alpha\alpha,\ \alpha\tau,\ \tau\tau,\ yy,\ y\beta,\ \beta\beta, \\ & xxx,\ xx\alpha,\ xx\tau,\ x\alpha\alpha,\ x\alpha\tau,\ x\tau\tau,\ xyy,\ xy\beta,\ x\beta\beta,\ \alpha\alpha\alpha, \\ & \alpha\alpha\tau,\ \alpha\tau\tau,\ \alpha yy,\ \alpha y\beta,\ \alpha\beta\beta,\ \tau\tau\tau,\ \tau yy,\ \tau y\beta,\ \tau\beta\beta, \end{aligned} \tag{6-13}$$

the explicit expressions of non-vanishing elements of $G(n|m)$ are :

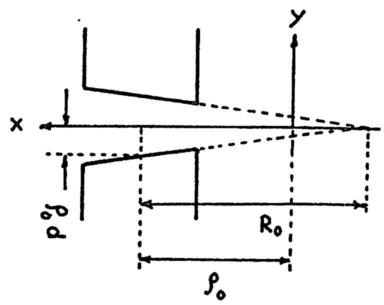


Fig.6. Cross section of magnet pole pieces.

$$G(\tau \mid 1) = \rho_0$$

$$G(xx \mid 1) = \frac{1}{\rho_0}\left(-\frac{k_x^2}{2} + q_1 \right)$$

$$G(xx \mid s_x s_x) = \frac{1}{\rho_0}\left[k_x^2 - q_1 \right]$$

$$G(x\alpha \mid s_x c_x) = -\frac{2\rho_0}{k_x} G(xx \mid s_x s_x)$$

$$G(x\tau \mid 1) = -\frac{2\rho_0}{k_x^2} G(xx \mid 1)$$

$$G(x\tau \mid c_x) = \left[k_x^2 + \frac{2q_1}{k_x^2} \right]$$

$$G(x\tau \mid s_x s_x) = -\frac{2\rho_0}{k_x^2} G(xx \mid s_x s_x)$$

$$G(\alpha\alpha \mid 1) = \frac{\rho_0}{2}$$

$$G(\alpha\alpha \mid s_x s_x) = -\frac{\rho_0^2}{k_x^2} G(xx \mid s_x s_x)$$

$$G(\alpha\tau \mid s_x) = \frac{\rho_0}{k_x} G(x\tau \mid c_x)$$

$$G(\alpha\tau \mid s_x c_x) = \frac{2\rho_0^2}{k_x^3} G(xx \mid s_x s_x)$$

$$G(\tau\tau \mid 1) = \frac{2\rho_0 q_1}{k_x^4}$$

$$G(\tau\tau \mid c_x) = -\frac{\rho_0}{k_x^2} G(x\tau \mid c_x)$$

$$G(\tau\tau \mid s_x s_x) = \frac{\rho_0^2}{k_x^4} G(xx \mid s_x s_x)$$

$$G(yy \mid 1) = \frac{1}{\rho_0}\left(\frac{n_1}{2} + n_2 \right)$$

$$G(yy \mid s_y s_y) = -\frac{n_2}{\rho_0}$$

$$G(y\beta \mid s_y c_y) = \left[\frac{2n_2}{k_y} \right]$$

$$G(\beta\beta|\ s_y s_y) = \frac{\rho_0 n_2}{k_y^2}$$

$$G(xxx|\ 1) = \frac{1}{\rho_0^2}[\ \frac{k_x^2}{3} + \frac{1}{3} - \frac{q_1}{3} - \frac{2q_1^2}{3k_x^2} - \frac{n_2}{3}\ ]$$

$$G(xxx|\ c_x) = \frac{1}{\rho_0^2}[\ \frac{k_x^4}{2} + \frac{q_1}{3} + \frac{2q_1^2}{3k_x^2} + q_2\ ]$$

$$G(xxx|\ s_x s_x) = \frac{1}{\rho_0^2}[\ \frac{k_x^2}{3} - \frac{q_1}{3} + \frac{2q_1^2}{3k_x^2}\ ]$$

$$G(xxx|s_x s_x c_x) = \frac{1}{\rho_0^2}[\ -k_x^4 + \frac{k_x^2}{3} - 2q_1 + \frac{2q_1^2}{3k_x^2} - q_2\ ]$$

$$G(xx\alpha|\ s_x) = \frac{1}{\rho_0}\ [\frac{3k_x^3}{2} - \frac{2k_x}{3} + \frac{13q_1}{3k_x} - \frac{2q_1^2}{3k_x^3} + \frac{3q_2}{k_x}\ ]$$

$$G(xx\alpha|\ s_x c_x) = \frac{1}{\rho_0}[\ \frac{k_x}{3} - \frac{q_1}{3k_x} + \frac{2q_1^2}{k_x^2}\ ]$$

$$G(xx\alpha|s_x s_x s_x) = \frac{3\rho_0}{k_x}\ G(xxx|s_x s_x c_x)$$

$$G(xx\tau|\ 1) = \frac{1}{\rho_0}[\ -1 + \frac{2k_x^2}{3} + \frac{q_1}{3} + \frac{4q_1}{3k_x^2} + \frac{8q_1^2}{3k_x^4} + \frac{3q_2}{k_x^2} + n_2\ ]$$

$$G(xx\tau|\ c_x) = \frac{1}{\rho_0}[\ 1 + 2k_x^2 + n_2\ ] - G(xx\tau|\ 1)$$

$$G(xx\tau|\ s_x s_x) = \frac{1}{\rho_0}[\ \frac{4k_x^2}{3} + 1 + \frac{q_1}{3} - \frac{5q_1}{3k_x^2} - \frac{4q_1^2}{3k_x^4} - \frac{3q_2}{k_x^2} - n_2\ ]$$

$$G(xx\tau|\ s_x c_x \omega) = \frac{1}{\rho_0}[\ -k_x^3 + k_x q_1 - \frac{2q_1}{k_x} + \frac{2q_1^2}{k_x^3}\ ]$$

$$G(xx\tau|s_x s_x c_x) = -\frac{3\rho_0}{k_x^2}\ G(xxx|s_x s_x c_x)$$

$$G(x\alpha\alpha|\ 1) = [\ \frac{1}{6} + \frac{5q_1}{3k_x^2} - \frac{4q_1^2}{3k_x^4}\ ]$$

$$G(x\alpha\alpha|\ c_x) = [\ -\frac{3k_x^2}{2} + \frac{1}{2}\ ] - G(x\alpha\alpha|\ 1)$$

$$G(x\alpha\alpha|\ s_x s_x) = [\ -\frac{1}{3} - \frac{2q_1}{k_x^2} + \frac{8q_1^2}{3k_x^4}\ ]$$

$$G(x\alpha\alpha|s_x s_x c_x) = -\frac{3\rho_0^2}{k_x^2}\ G(xxx|s_x s_x c_x)$$

$$G(x\alpha\tau|\quad\omega) = -\frac{\rho_0}{k_x} G(xx\tau|s_x c_x\omega)$$

$$G(x\alpha\tau|\quad s_x) = [\ -\frac{8k_x}{3} + \frac{4}{3k_x} + \frac{2q_1}{3k_x} - \frac{8q_1}{k_x^3} + \frac{8q_1^2}{3k_x^5} - \frac{6q_2}{k_x^3}\ ]$$

$$G(x\alpha\tau|\quad s_x c_x) = [\ \frac{5k_x}{3} - \frac{4}{3k_x} + \frac{q_1}{3k_x} + \frac{6q_1}{k_x^3} - \frac{2q_1^2}{3k_x^5} + \frac{6q_2}{k_x^3} + \frac{2n_2}{k_x}\ ]$$

$$G(x\alpha\tau|\ s_x s_x\omega) = \frac{2\rho_0}{k_x} G(xx\tau|\ s_x c_x\omega)$$

$$G(x\alpha\tau|s_x s_x s_x) = \frac{-6\rho_0^2}{k_x^3} G(xxx|s_x s_x c_x)$$

$$G(x\tau\tau|\quad 1) = [\ -\frac{4}{3} + \frac{2}{k_x^2} - \frac{2q_1}{3k_x^2} - \frac{8q_1}{3k_x^4} - \frac{16q_1^2}{3k_x^6} - \frac{6q_2}{k_x^4} - \frac{2n_2}{k_x^2}\ ]$$

$$G(x\tau\tau|\quad c_x) = [\ -1 - \frac{1}{k_x^2}\ ] - G(x\tau\tau|\quad 1)$$

$$G(x\tau\tau|\quad s_x\omega\ ) = [\ \frac{k_x^3}{2} + \frac{2q_1}{k_x} + \frac{2q_1^2}{k_x^5}\ ]$$

$$G(x\tau\tau|\quad s_x s_x) = [\ \frac{2}{3} - \frac{2}{k_x^2} - \frac{2q_1}{3k_x^2} + \frac{10q_1}{3k_x^4} + \frac{8q_1^2}{3k_x^6} + \frac{6q_2}{k_x^4} + \frac{2n_2}{k_x^2}\ ]$$

$$G(x\tau\tau|\ s_x c_x\omega) = -\frac{2\rho_0}{k_x^2} G(xx\tau|\ s_x c_x\omega)$$

$$G(x\tau\tau|s_x s_x c_x) = \frac{3\rho_0^2}{k_x^4} G(xxx|s_x s_x c_x)$$

$$G(xyy|\quad 1) = \frac{1}{\rho_0^2}[\ \frac{n_1}{2} + n_2 - q_1(\ \frac{n_2}{q_7} + \frac{n_2-k_y^2}{k_x^2}\ ) + \frac{4n_2^2}{q_7}\ ]$$

$$G(xyy|\quad c_x) = \frac{1}{\rho_0^2}[\ \frac{n_1+n_1^2}{2} + (\ \frac{n_2}{q_7} + \frac{n_1+n_2}{k_x^2}\ )q_1 - \frac{4n_2^2}{q_7} + q_3\ ]$$

$$G(xyy|\quad s_x s_x) = \frac{1}{\rho_0^2}[\ (q_1 - k_x^2)(\frac{n_2}{q_7} + \frac{n_2-k_y^2}{k_x^2})\ ]$$

$$G(xyy|\quad s_y s_y) = \frac{1}{\rho_0^2}[\ -\frac{4n_2^2}{k_y^2 q_7}\ ]$$

$$G(xyy|s_y s_y c_y) = \frac{1}{\rho_0^2}[\ k_x k_y n_1 + 2k_y(\frac{2k_x}{q_7} - \frac{1}{k_x})n_2 - \frac{8k_y n_2^2}{k_x q_7}\ ]$$

$$G(xyy|c_x s_y s_y) = \frac{1}{\rho_0^2}[\ -\frac{2n_2 q_1}{q_7} + \frac{4n_2^2}{q_7} - q_3\ ]$$

$$G(xy\beta|\quad s_x) = -\frac{\rho_0}{k_y} G(xyy|s_x s_y c_y)$$

$$G(xy\beta|\quad s_x c_x) = \frac{1}{\rho_0}\left[\frac{4n_2(q_1 - k_x^2)}{k_x q_7} \right]$$

$$G(xy\beta|s_x s_y s_y) = \frac{2\rho_0}{k_y} G(xyy|s_x s_y c_y)$$

$$G(xy\beta|c_x s_y c_y) = -\frac{2\rho_0}{k_y} G(xyy|c_x s_y s_y)$$

$$G(x\beta\beta|\quad 1) = \left[-\frac{1}{2} + \frac{q_1}{k_x^2} + \frac{4n_2 q_1}{k_x^2 q_7} \right]$$

$$G(x\beta\beta|\quad c_x) = \left[-\frac{k_x^2}{2} - \frac{1}{2} \right] - G(x\beta\beta|\quad 1)$$

$$G(x\beta\beta|\quad s_x s_x) = \left[\left(1 - \frac{q_1}{k_x^2}\right)\left(1 + \frac{4n_2}{q_7}\right) \right]$$

$$G(x\beta\beta|\quad s_y s_y) = \left[-\frac{4n_2^2}{k_y^2 q_7} \right]$$

$$G(x\beta\beta|s_x s_y c_y) = -\frac{\rho_0^2}{k_y^2} G(xyy|s_x s_y c_y)$$

$$G(x\beta\beta|c_x s_y s_y) = -\frac{\rho_0^2}{k_y^2} G(xyy|c_x s_y s_y)$$

$$G(\alpha\alpha\alpha|\quad s_x) = \frac{\rho_0}{k_x} G(x\alpha\alpha|\quad c_x)$$

$$G(\alpha\alpha\alpha|\quad s_x c_x) = \rho_0\left[-\frac{1}{3} + \frac{5q_1}{3k_x^2} - \frac{4q_1^2}{3k_x^4} \right]$$

$$G(\alpha\alpha\alpha|s_x s_x s_x) = -\frac{\rho_0^3}{k_x^3} G(xxx|s_x s_x c_x)$$

$$G(\alpha\alpha\tau|\quad 1) = \rho_0\left[-\frac{7}{6} + \frac{1}{3k_x^2} + \frac{2q_1}{3k_x^2} - \frac{2q_1}{k_x^4} + \frac{8q_1^2}{3k_x^6} \right]$$

$$G(\alpha\alpha\tau|\quad c_x) = \left[\frac{\rho_0}{2} \right] - G(\alpha\alpha\tau|\quad 1)$$

$$G(\alpha\alpha\tau|\quad s_x s_x) = \rho_0\left[\frac{1}{3} - \frac{5}{3k_x^2} + \frac{2q_1}{3k_x^2} + \frac{5q_1}{k_x^4} - \frac{4q_1^2}{3k_x^6} + \frac{3q_2}{k_x^4} + \frac{n_2}{k_x^2} \right]$$

$$G(\alpha\alpha\tau|\ s_x c_x \omega) = -\frac{\rho_0^2}{k_x^2} G(xx\tau|\ s_x c_x \omega)$$

$$G(\alpha\alpha\tau|s_x s_x c_x) = \frac{3\rho_0^3}{k_x^4} G(xxx|s_x s_x c_x)$$

$$G(\alpha\tau\tau|\quad \omega) = \frac{\rho_0^2}{k_x^3} G(xx\tau|s_x c_x\ \omega)$$

$$G(\alpha\tau\tau|\quad s_x) = \left[-\frac{k_x}{2} + \frac{11}{3k_x} - \frac{8}{3k_x^3} + \frac{4q_1}{3k_x^3} + \frac{20q_1}{3k_x^5} + \frac{2q_1^2}{k_x^7} + \frac{6q_2}{k_x^5} + \frac{2n_2}{k_x^3} \right]$$

$$G(\alpha\tau\tau|\ c_x\ ) = -\frac{\rho_0}{k_x} G(x\tau\tau|\ s_x\ \omega)$$

$$G(\alpha\tau\tau|\ s_x c_x) = \rho_0 \left[-\frac{5}{3k_x} + \frac{8}{3k_x^3} - \frac{q_1}{3k_x^3} - \frac{14q_1}{3k_x^5} - \frac{2q_1^2}{k_x^7} - \frac{6q_2}{k_x^5} - \frac{2n_2}{k_x^3} \right]$$

$$G(\alpha\tau\tau|s_x s_x\ ) = -\frac{\rho_0^2}{k_x^3} G(xx\tau|s_x c_x\ \omega)$$

$$G(\alpha\tau\tau|s_x s_x s_x) = \frac{3\rho_0^3}{k_x^5} G(xxx|s_x s_x c_x)$$

$$G(\alpha yy|\quad s_x) = \frac{\rho_0}{k_x} G(xyy|\quad c_x)$$

$$G(\alpha yy|\ s_x c_x) = \frac{1}{\rho_0}\left[-\frac{1}{2k_x}\left(\frac{n_2}{q_7} + \frac{n_2 - k_y^2}{k_x^2} \right) \right]$$

$$G(\alpha yy|\ s_y c_y) = \frac{1}{\rho_0}\left[\frac{2n_2}{k_x^2}\left(\frac{2q_6 n_2}{k_y q_7} - k_y \right) \right]$$

$$G(\alpha yy|s_x s_y s_y) = \frac{\rho_0}{k_x} G(xyy|c_x s_y s_y)$$

$$G(\alpha yy|c_x s_y c_y) = -\frac{\rho_0}{k_x} G(xyy|s_x s_y c_y)$$

$$G(\alpha y\beta|\quad 1) = \left[\left(\frac{2}{k_x^2} - \frac{4}{q_7}\right) n_2 + \frac{8n_2^2}{k_x^2 q_7} \right]$$

$$G(\alpha y\beta|\quad c_x) = \frac{\rho_0^2}{k_x k_y} G(xyy|s_x s_y c_y)$$

$$G(\alpha y\beta|\ s_x s_x) = \frac{\rho_0}{k_x} G(xy\beta|\ s_x c_x)$$

$$G(\alpha y\beta|\ s_y s_y) = \left[-\frac{n_2}{k_x^2} + \frac{4n_2^2}{k_x^2 k_y^2} \right]$$

$$G(\alpha y\beta|s_x s_y c_y) = -\frac{2\rho_0^2}{k_x k_y} G(xyy|c_x s_y s_y)$$

$$G(\alpha y\beta | c_x s_y s_y) = \frac{2\rho_0^2}{k_x k_y} G(xyy | s_x s_y c_y)$$

$$G(\alpha\beta\beta | \quad s_x) = \rho_0 [\frac{k_x}{2} - \frac{q_1}{k_x^3} - \frac{4n_2 q_1}{k_x^3 q_7}]$$

$$G(\alpha\beta\beta | \quad s_x c_x) = \rho_0 [\frac{1}{k_x^3} (\frac{1}{2} + \frac{2n_2}{q_7})]$$

$$G(\alpha\beta\beta | \quad s_y c_y) = \rho_0 [\frac{2n_2}{k_x^2 k_y} + \frac{8n_2^2}{k_x^2 k_y q_7}]$$

$$G(\alpha\beta\beta | s_x s_y c_y) = \frac{\rho_0^3}{k_x k_y^2} G(xyy | c_x s_y s_y)$$

$$G(\alpha\beta\beta | c_x s_y c_y) = \frac{\rho_0^3}{k_x k_y^2} G(xyy | s_x s_y c_y)$$

$$G(\tau\tau\tau | \quad 1) = \rho_0 [\frac{5}{3k_x^2} - \frac{2}{k_x^4} + \frac{4q_1}{3k_x^4} + \frac{8q_1}{3k_x^6} + \frac{16q_1^2}{3k_x^8} + \frac{4q_2}{k_x^6} + \frac{2n_2}{k_x^4}]$$

$$G(\tau\tau\tau | \quad c_x) = \rho_0 [1 + \frac{4q_2}{k_x^6} + \frac{2n_2}{k_x^4}] - G(\tau\tau\tau | \quad 1)$$

$$G(\tau\tau\tau | \quad s_x \omega) = \frac{\rho_0}{k_x^2} G(x\tau\tau | \quad s_x \omega)$$

$$G(\tau\tau\tau | \quad s_x s_x) = \rho_0 [\frac{4}{3k_x^4} - \frac{4}{3k_x^2} + \frac{q_1}{3k_x^4} - \frac{4q_1}{3k_x^6} + \frac{2q_1^2}{k_x^8} - \frac{3q_2}{k_x^6} - \frac{n_2}{k_x^4}]$$

$$G(\tau\tau\tau | s_x c_x \quad) = \frac{\rho_0^2}{k_x^4} G(xx\tau | s_x c_x \omega)$$

$$G(\tau\tau\tau | s_x s_x c_x) = \frac{\rho_0^3}{k_x^6} G(xxx | s_x s_x c_x)$$

$$G(\tau yy | \quad 1) = \frac{1}{\rho_0} [\frac{k_x^2}{2} (1 + \frac{4q_1}{k_x^4}) (\frac{n_2}{q_7} + \frac{n_2 - k_y^2}{k_x^2}) - \frac{4n_2^2}{k_x^2 q_7} + \frac{q_3}{k_x^2} + \frac{n_1 + n_1^2}{2k_x^2} - n_2 - \frac{n_1}{2}]$$

$$G(\tau yy | \quad c_x) = \frac{1}{\rho_0} [\frac{n_1}{2} - n_2] - G(\tau yy | \quad 1)$$

$$G(\tau yy | \quad s_x s_x) = \frac{\rho_0}{k_x^2} G(xyy | \quad s_x s_x)$$

$$G(\tau yy | \quad s_y s_y) = \frac{1}{\rho_0} [\frac{n_1}{2} - \frac{4k_y^2 n_2}{q_7} - \frac{2n_2 q_1}{k_x^2 q_7} + \frac{4n_2^2}{k_x^2 q_7} - \frac{q_3}{k_x^2}]$$

$$G(\tau yy | s_y c_y \omega) = \frac{1}{\rho_0} [k_y n_2 + \frac{2n_2 q_6}{k_x^2 k_y}]$$

$$G(\tau yy|s_x s_y c_y) = -\frac{\rho_0}{k_x^2} G(xyy|s_x s_y c_y)$$

$$G(\tau yy|c_x s_y s_y) = -\frac{\rho_0}{k_x^2} G(xyy|c_x s_y s_y)$$

$$G(\tau y\beta|\quad \omega) = [\ -\frac{2n_2}{k_y^2}\left(\frac{n_1+n_2}{k_x^2} - \frac{n_1}{2}\right)\ ]$$

$$G(\tau y\beta|\quad s_x) = [\ \frac{n_1}{k_x} + \left(\frac{4}{k_x q_7} - \frac{2k_x}{q_7} - \frac{2}{k_x^3}\right) n_2 - \frac{4n_2 q_1}{k_x^3 q_7} - \frac{8n_2^2}{k_x^3 q_7}\ ]$$

$$G(\tau y\beta|\quad s_x c_x) = -\frac{\rho_0}{k_x^2} G(xy\beta|\quad s_x c_x)$$

$$G(\tau y\beta|\quad s_y c_y) = [\ \frac{n_1-n_2}{k_y} + \left(\frac{2q_1}{k_x^2} + k_x^2\right)\frac{2n_2}{k_y q_7} + \frac{2q_3}{k_x^2 k_y} + \frac{2n_2^2}{k_x^2 k_y^3}\ ]$$

$$G(\tau y\beta|s_y s_y\ \omega) = [\ \left(2 - \frac{4}{k_x^2}\right) n_2 + \frac{4n_2^2}{k_x^2 k_y^2}\ ]$$

$$G(\tau y\beta|s_x s_y s_y) = \frac{2\rho_0^2}{k_x^2 k_y} G(xyy|s_x s_y c_y)$$

$$G(\tau y\beta|c_x s_y c_y) = \frac{2\rho_0^2}{k_x^2 k_y} G(xyy|c_x s_y s_y)$$

$$G(\tau\beta\beta|\quad 1) = \rho_0 [\ -\frac{3}{2} - \frac{2q_1}{k_x^4} - \frac{8n_2 q_1}{k_x^4 q_7} - \frac{2n_2}{q_7}\ ]$$

$$G(\tau\beta\beta|\quad c_x) = -\ G(\tau\beta\beta|\quad 1)$$

$$G(\tau\beta\beta|\quad s_x s_x) = -\frac{\rho_0}{k_x^2} G(x\beta\beta|\quad s_x s_x)$$

$$G(\tau\beta\beta|\quad s_y s_y) = \rho_0 [\ \frac{n_1}{2k_y^2} + \frac{k_x^2 n_2}{k_y^2 q_7} + \frac{2n_2 q_1}{k_x^2 k_y^2 q_7} + \frac{2q_6 n_2^2}{k_x^2 k_y^4 q_7} + \frac{q_3}{k_x^2 k_y^2}\ ]$$

$$G(\tau\beta\beta|s_y c_y\ \omega) = -\frac{\rho_0^2}{k_y^2} G(\tau yy|s_x c_y\ \omega)$$

$$G(\tau\beta\beta|s_x s_y c_y) = \frac{\rho_0^3}{k_x^2 k_y^2} G(xyy|s_x s_y c_y)$$

$$G(\tau\beta\beta|c_x s_y s_y) = \frac{\rho_0^3}{k_x^2 k_y^2} G(xyy|c_x s_y s_y) \quad .$$

(6-14)

The solution $x(\omega)$ is also considered to be a series of m and n

$$x(\omega) = \sum_m m \sum_n n \cdot H(n|m) \tag{6-15}$$

The relationship between $H(n|m)$ and $G(n|m)$ is given by eqs.(5-34) for the electric field. The expression (6-15) of $x(\omega)$ can be rearranged as

$$x(\omega) = \sum_n n \sum_m m\, H(n|m) \tag{6-16}$$

In this expression the coefficients of n give the elements of the first row of the transfer matrix. The inclination angle α is also derived in the same way as in electric field,

$$\tan\alpha = \sum_n n \sum_m m[H_1'(n|m) + H_2'(n|m)] \quad . \tag{6-17}$$

The matrix elements of H' and H' are the same as those of eqs.(5-38). The coefficients of n of eq.(6-17) give the elements of the second row of the transfer matrix. Then we can construct the third order radial transfer matrix in an inhomogeneous magnetic field. Furthermore the explicit expressions of the elements of transfer matrix are given in ref.(9).

6.2.2 Axial Direction

The solution for y direction is obtained in a way similar to x direction. From eq.(6-9b) we obtain

$$\frac{d^2y}{d\omega^2} + k_y^2 y = \sum_m m \sum n\, G(n|m) \tag{6-18}$$

where in this case m and n are the components of the following vectors :

$$m = s_y,\ c_y,\ s_y\omega,\ c_y\omega,\ s_x s_y,\ s_x c_y,\ c_x s_y,\ c_x c_y, \tag{6-19}$$

$$n = y,\ \beta,\ yx,\ x\alpha,\ x\tau,\ \beta x,\ \beta\alpha,\ \beta\tau, \tag{6-20}$$

The nonvanishing elements of $G(n|m)$ are as follows :

$$G(yx|s_x s_y) = \frac{k_x k_y}{\rho_0}$$

$$G(yx|c_x c_y) = \frac{2(n_1+n_2)}{\rho_0}$$

$$G(y\alpha|s_x c_y) = \frac{n_1+n_2}{2k_x}$$

$$G(y\alpha|c_x s_y) = -k_y$$

$$G(y\tau|\;\;c_y) = \frac{2(n_1+n_2)}{k_x^2} - n_1$$

$$G(y\tau|s_xs_y) = -\frac{k_y}{k_x}$$

$$G(y\tau|c_xs_y) = -\frac{2(n_1+n_2)}{k_x^2}$$

$$G(\beta x|s_xc_y) = -k_x$$

$$G(\beta x|c_xs_y) = \frac{2(n_1+n_2)}{k_y}$$

$$G(\beta\alpha|s_xc_y) = \frac{2\rho_0(n_1+n_2)}{k_xk_y}$$

$$G(\beta\alpha|c_xc_y) = \rho_0$$

$$G(\beta\tau|\;\;s_y) = \rho_0\left[\frac{2(n_1+n_2)}{k_x^2k_y} - \frac{n_1}{k_y}\right]$$

$$G(\beta\tau|s_xc_y) = \frac{\rho_0}{k_x}$$

(6-21)

$$G(\beta\tau|c_xs_y) = -\frac{2\rho_0(n_1+n_2)}{k_x^2k_y} \quad . \qquad (6\text{-}21)$$

The solution of eq.(6-18) is assumed to be

$$y(\omega) = \sum_m n \sum_n m\; H(n|m) \quad . \qquad (6\text{-}22)$$

The matrix elements of $H(n|m)$ are given in eq.(5-25b) .
Similar to eq.(6-17) angle $\beta(\omega)$ is written as

$$\beta(\omega) = \sum_n n \sum_m m\; [\; H_1^{\beta}(n|m) + H_2^{\beta}(n|m)\;] \quad . \qquad (6\text{-}23)$$

The nonvanishing elements of H_1^{β} and H_2^{β} are same as those of eqs.(5-28b).

[7] Third Order Image Aberration

7.1. Third Order Image Aberration of Single Field Spectrometer

In chapters [5], [6] we obtained the third order transfer matrices in electric field and magnetic field. If the spectrometer consists of a single field, that is, only electric field or magnetic field, the third order image aberration can easily be derived. According to the procedure stated in chapter [3], by multiplication of each transfer matrix of drift space, finging field[6)] and main field, we can determine the total overall transfer matrix ${}^{3}_{x}T_{i}$ and ${}^{2}_{y}T_{i}$. Then the transformation between the initial position vector and the final position vector is written in the following form :

$$
\begin{pmatrix} x \\ \alpha \\ \delta \\ xx \\ \cdot \\ \cdot \\ xxx \\ \cdot \\ \cdot \\ \cdot \\ \delta\beta\beta \end{pmatrix}
=
\begin{pmatrix}
{}_{x}T_{1,1} & {}_{x}T_{1,2} & \cdot & \cdot & \cdot & {}_{x}T_{1,31} \\
{}_{x}T_{2,1} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & {}_{x}T_{31,31}
\end{pmatrix}
\begin{pmatrix} x \\ \alpha \\ \delta \\ xx \\ \cdot \\ \cdot \\ xxx \\ \cdot \\ \cdot \\ \cdot \\ \delta\beta\beta \end{pmatrix}
$$

$$
\begin{pmatrix} y \\ \beta \\ yx \\ \cdot \\ \cdot \\ \cdot \\ \beta\delta \end{pmatrix}
=
\begin{pmatrix}
{}_{y}T_{1,1} & {}_{y}T_{1,2} & \cdot & \cdot & \cdot & {}_{y}T_{1,8} \\
{}_{y}T_{2,1} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & {}_{y}T_{8,8}
\end{pmatrix}
\begin{pmatrix} y \\ \beta \\ yx \\ \cdot \\ \cdot \\ \cdot \\ \beta\delta \end{pmatrix}
\quad (7\text{-}1a)
$$

We write the above relationship in short form as :

$$ {}^{3}_{x}V_{f} = {}^{3}_{x}T \cdot {}^{3}_{x}V_{i} \tag{7-2a}$$

$$ {}^{2}_{y}V_{f} = {}^{2}_{y}T \cdot {}^{2}_{y}V_{i} \tag{7-2b}$$

where suffix i and

purpose of getting high resolving power, not only the first order direction focusing but also the second order focusing must be satisfied. Necessary geometrical parameters should be chosen so that the coefficient ${}_{x}T_{1,2}$ of eq.(7-1a) equals to zero (first order focusing condition) and ${}_{x}T_{1,5}$, ${}_{x}T_{1,7}$, ${}_{x}T_{1,8}$ as small as possible.

For assuming the total image aberration to be small as expected we must estimate the magnitude of the third order contributions. It is easily possible to execute such calculation by only introducing the parameter into each transfer matrix and obtaining the expression of eq.(7-1a) and (7-1b). Then the coefficients ${}_{x}T_{1,13}$, ${}_{x}T_{1,14}$ ..... ${}_{x}T_{1,31}$ give the third order coefficients wanted.

7.2. Third Order Image Aberration of Double Focusing Spectrometer Consisting of Electric and Magnetic Fields

Independent elements of position vector are in electric field (displacement x, y; angle α, β; mass γ; energy δ) and in magnetic field (displacement x, y; angle α, β; mass γ; momentum τ). Therefore when taking both electric and magnetic field, it seems necessary to use seven elements (x, y, α, β, γ, δ, τ). There exists, however, the following relation among mass, energy and momentum.

$$\delta = -\gamma + 2\tau + \gamma^2 - 2\gamma\tau + \tau^2 - \gamma^3 + 2\gamma^2\tau - \gamma\tau^2$$

$$\tau = \frac{1}{2}(\gamma+\delta) - \frac{1}{8}(\gamma^2-2\gamma\delta+\delta^2) + \frac{1}{16}(\gamma^3-\gamma^2\delta-\gamma\delta^2+\delta^3) \tag{7-3}$$

Then it is sufficient to introduce six independent elements. In the case of mass spectrometer it is possible to choose the velocity deviation as one of the independent elements instead of energy deviation. And many authers have used this expression till now. Since ions are accelerated under the condition of constant voltage it is much advantageous to choose the energy deviation as an independent element in higher order calculation. From now we choose 6 independent elements (x, y, α, β, γ, δ) in the system which consists of electric and magnetic field.

On the contrary, the transfer matrix of the magnetic field obtained in chapter [6] contains the momentum deviation τ instead of energy δ. Then it is necessary to know the relationship for the transformation between the position vector including (γ,δ) and that including τ. The position vector including τ is given by 31 elements as :

$$(x,\ \alpha,\ \tau,\ xx,\ x\alpha,\ x\tau,\ \alpha\alpha,\ \alpha\tau,\ \tau\tau,\ yy,\ y\beta,\ \beta\beta,\ xxx,\ xx\alpha,\ xx\tau,\ x\alpha\alpha,\ x\alpha\tau,\ x\tau\tau,\ xyy,\ xy\beta,\ x\beta\beta,\ \alpha\alpha\alpha,\ \alpha\alpha\tau,\ \alpha\tau\tau,\ \alpha yy,\ \alpha y\beta,\ \alpha\beta\beta,\ \tau\tau\tau,\ \tau yy,\ \tau y\beta,\ \tau\beta\beta) \qquad (7\text{-}4)$$

While the position vector including γ and δ is given by 49 elements as :

$$(x,\ \alpha,\ \gamma,\ \delta,\ xx,\ x\alpha,\ x\gamma,\ x\delta,\ \alpha\alpha,\ \alpha\gamma,\ \alpha\delta,\ \gamma\gamma,\ \gamma\delta,\ \delta\delta,\ yy,\ y\beta,\ \beta\beta,\ xxx,\ xx\alpha,\ xx\gamma,\ xx\delta,\ x\alpha\alpha,\ x\alpha\gamma,\ x\alpha\delta,\ x\gamma\gamma,\ x\gamma\delta,\ x\delta\delta,\ xyy,\ xy\beta,\ x\beta\beta,\ \alpha\alpha\alpha,\ \alpha\alpha\gamma,\ \alpha\alpha\delta,\ \alpha\gamma\gamma,\ \alpha\gamma\delta,\ \alpha\delta\delta,\ \alpha yy,\ \alpha y\beta,\ \alpha\beta\beta,\ \gamma\gamma\gamma,\ \gamma\gamma\delta,\ \gamma\delta\delta,\ \gamma yy,\ \gamma y\beta,\ \gamma\beta\beta,\ \delta\delta\delta,\ \delta yy,\ \delta y\beta,\ \delta\beta\beta) \qquad (7\text{-}5)$$

The first step to do is to extend the number of elements from (31 x 31) to (49 x 49). This is easily done by rearranging the elements of (31 x 31) matrix into the (49 x 49) matrix. In this case τ coresponds to δ and elements coresponding to γ are put to be zero. In the next place we define a new (49 x 49) matrix "Element Change Matrix(Cl)" which transform the elements of the position vector (γ,δ) to that of (γ,τ) and give no other change about the position and the inclination angle of the beam. The non-zero elements of this matrix are :

$R_{4,3} = \frac{1}{2}$,	$R_{8,25} = -\frac{1}{8}$,	$R_{13,41} = \frac{1}{4}$,	$R_{24,23} = \frac{1}{2}$,	$R_{36,34} = \frac{1}{4}$,	$R_{46,46} = \frac{1}{8}$
$R_{4,4} = \frac{1}{2}$,	$R_{8,26} = \frac{1}{4}$,	$R_{13,42} = -\frac{1}{8}$,	$R_{24,24} = \frac{1}{2}$,	$R_{36,35} = \frac{1}{2}$,	$R_{47,43} = \frac{1}{2}$
$R_{4,12} = -\frac{1}{8}$,	$R_{8,27} = -\frac{1}{8}$,	$R_{14,12} = \frac{1}{4}$,	$R_{26,25} = \frac{1}{2}$,	$R_{36,36} = \frac{1}{4}$,	$R_{47,47} = \frac{1}{2}$
$R_{4,13} = \frac{1}{4}$,	$R_{11,10} = \frac{1}{2}$,	$R_{14,13} = \frac{1}{2}$,	$R_{26,26} = \frac{1}{2}$,	$R_{41,40} = \frac{1}{2}$,	$R_{48,44} = \frac{1}{2}$
$R_{4,14} = -\frac{1}{8}$,	$R_{11,11} = \frac{1}{2}$,	$R_{14,14} = \frac{1}{4}$,	$R_{27,25} = \frac{1}{4}$,	$R_{41,41} = \frac{1}{2}$,	$R_{48,48} = \frac{1}{2}$
$R_{4,40} = \frac{1}{16}$,	$R_{11,34} = -\frac{1}{8}$,	$R_{14,40} = -\frac{1}{8}$,	$R_{27,26} = \frac{1}{2}$,	$R_{42,40} = \frac{1}{4}$,	$R_{49,46} = \frac{1}{2}$
$R_{4,41} = -\frac{1}{16}$,	$R_{11,35} = \frac{1}{4}$,	$R_{14,41} = -\frac{1}{8}$,	$R_{27,27} = \frac{1}{4}$,	$R_{42,41} = \frac{1}{2}$,	$R_{49,49} = \frac{1}{2}$
$R_{4,42} = -\frac{1}{16}$,	$R_{11,36} = -\frac{1}{8}$,	$R_{14,42} = -\frac{1}{8}$,	$R_{33,32} = \frac{1}{2}$,	$R_{42,42} = \frac{1}{4}$,	
$R_{4,46} = \frac{1}{16}$,	$R_{13,12} = \frac{1}{2}$,	$R_{14,46} = -\frac{1}{8}$,	$R_{33,33} = \frac{1}{2}$,	$R_{46,40} = \frac{1}{8}$,	
$R_{8,7} = \frac{1}{2}$,	$R_{13,13} = \frac{1}{2}$,	$R_{21,20} = \frac{1}{2}$,	$R_{35,34} = \frac{1}{2}$,	$R_{46,41} = \frac{3}{8}$,	
$R_{8,8} = \frac{1}{2}$,	$R_{13,40} = -\frac{1}{8}$,	$R_{21,21} = \frac{1}{2}$,	$R_{35,35} = \frac{1}{2}$,	$R_{46,42} = \frac{3}{8}$,	(7-6)

Other diagonal elements should equal to unity. We must define another element change matrix (C2) which transform the elements of position vector (γ,τ) to that of (γ,δ) this time.

The non-zero elements of this matrix are:

$R_{4,3} = -1$,	$R_{11,10} = -1$,	$R_{14,40} = -2$,	$R_{33,32} = -1$,	$R_{46,41} = 6$,
$R_{4,4} = 2$,	$R_{11,10} = 2$,	$R_{14,41} = 8$,	$R_{33,33} = 2$,	$R_{46,42} = -12$,
$R_{4,12} = 1$,	$R_{11,34} = 1$,	$R_{14,42} = -10$,	$R_{35,34} = -1$,	$R_{46,46} = 8$,
$R_{4,13} = -2$,	$R_{11,35} = -2$,	$R_{14,46} = 4$,	$R_{35,35} = 2$,	$R_{47,43} = -1$,
$R_{4,14} = 1$,	$R_{11,36} = 1$,	$R_{21,20} = -1$,	$R_{36,34} = 1$,	$R_{47,47} = 2$,
$R_{4,40} = -1$,	$R_{13,12} = -1$,	$R_{21,21} = 2$,	$R_{36,35} = -4$,	$R_{48,44} = -1$,
$R_{4,41} = 2$,	$R_{13,13} = 2$,	$R_{24,23} = -1$,	$R_{36,36} = 4$,	$R_{48,48} = 2$
$R_{4,42} = -1$,	$R_{13,40} = 1$,	$R_{24,24} = 2$	$R_{41,40} = -1$,	$R_{49,45} = -1$,
$R_{8,7} = -1$,	$R_{13,41} = -2$,	$R_{26,25} = -1$,	$R_{41,41} = 2$,	$R_{49,49} = 2$,
$R_{8,8} = 2$,	$R_{13,41} = 1$,	$R_{26,26} = 2$,	$R_{42,40} = 1$,	
$R_{8,25} = 1$,	$R_{14,12} = 1$,	$R_{27,25} = 1$,	$R_{42,41} = -4$,	
$R_{8,26} = -2$,	$R_{14,13} = -4$,	$R_{27,26} = -4$,	$R_{42,42} = 4$,	(7-7)
$R_{8,27} = 1$,	$R_{14,14} = 4$,	$R_{27,27} = 4$,	$R_{46,46} = -1$,	

And other diagonal elements should equal to unity. As a result the overall transfer matrix is expressed in the following form :

$$(T) = (L3)(C2)(M)(C1)(L2)(E)(L1) \qquad (7\text{-}8)$$

where (L1), (L2) and (L3) are drift spaces, (E) electric field and (M) magnetic field. (C1) and (C2) are element change matrices stated above. The following procedure is same as discussed in section (7.1.). When the system consists of many field, we can execute the calculations by repeating the same procedure.

[8] Second Order Transfer Matrix along the Non-Circular Main Path

In the design of particle spectrometers of high resolution image aberrations of second order are of high importance. So far such image aberrations have been derived using the calculations of particle trajectories relative to a circular main path. When we perform experiments of passing the ion beam, we often find the fact that the main beam does not pass along the circular main path which coincides with geometrical center of the instrument, but along a non-circular main path, and also the fact that in some cases by letting the main path deviate slightly from the originally designed circular one, the better direction and energy focusing can be achieved.

Then it would be desirable to describe such trajectories relative to a non circular main path which furthermore should be permitted to cross obliquely the ideal field boundaries of the magnetic and/or electrostatic sector field. In order to describe an arbitrary ray referring to a non circular main path, we first describe the same ray referring to the geometrical circular main path.

In this case the envelope of the particle beam would exceed the limits of calculations of normal second order approximation. Therefore it is necessary to use the third order calculations. After this calculation we transform the results to the new expressions relative to the non-circular main path. This method is investigated in the following section theoretically.

8.1. Particle Trajectories Relative to a New Non-Circular Main Path

In chapters [5], [6] we have defined the transfer matrix and position vector. There are following relations between them.

$$
\begin{aligned}
{}^3_x V_{i+1} &= {}^3_x T_i \cdot {}^3_x V_i \\
{}^2_y V_{i+1} &= {}^2_y T_i \cdot {}^2_y V_i
\end{aligned}
\qquad (8\text{-}1)
$$

where

$${}^3_x V_i = (x,\ \alpha,\ \delta,\ xx \cdots xxx \cdots \delta\beta\beta)_i$$

$${}^2_y V_i = (y,\ \beta,\ yx,\ \cdots\ \beta\delta\ )_i$$

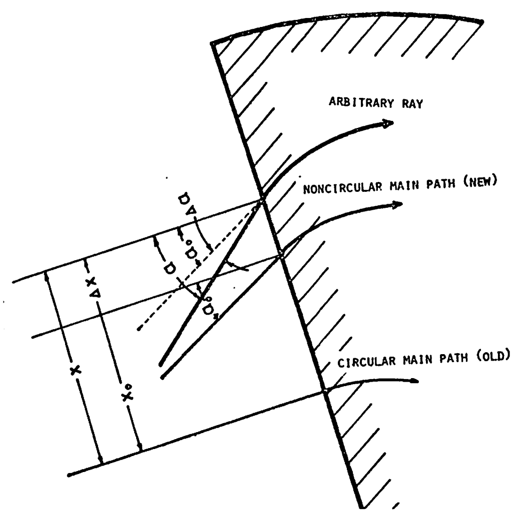
$${}^3_x T_i = \begin{pmatrix} {}_xT_{11} & {}_xT_{12} & \cdots \\ {}_xT_{21} & & \\ \vdots & & \end{pmatrix}_i$$

$$ {}^{2}_{y}T_i = \begin{pmatrix} {}_{y}T_{11} & {}_{y}T_{12} & \cdots \\ {}_{y}T_{21} & & \\ \vdots & & \end{pmatrix}_i \qquad (8\text{-}2) $$

In order to change the main path from circular to non circular we have to change the coordinateds x, α, y, β, δ as follows.

$$ \begin{aligned} x &\to x_0 + \Delta x \\ \alpha &\to \alpha_0 + \Delta\alpha \\ \delta &\to \delta_0 + \Delta\delta \\ y &\to y_0 + \Delta y \\ \beta &\to \beta_0 + \Delta\beta \end{aligned} \qquad (8\text{-}3) $$

THE COORDINATE SYSTEM OF NONCIRCULAR MAIN PATH



Here the x , y , α, β describe the new main path relative to the old circular one and the Δx, Δy, $\Delta\alpha$, $\Delta\beta$ describe the particle trajectory relative to the new main path. The δ_0 characterizes the change in energy or momentum of a particle moving along the new main path relative to one moving along the old main path. Thus $\Delta\delta$ finally describes the energy or momentum deviation of any particle from a particle moving along the new main path. The value of δ_0 is normally given by changing the accelerating voltage, however, it can be controled also by changing the field strength as described in [8.2.2.]

Let us introduce now eqs.(8-3) into vectors ${}^{3}_{x}V$ and ${}^{2}_{y}V$ of eq.(8-2). In this case eqs.(8-3) describe in an involved manner the position of the arbitrary beam relative to the old circular one as well as to the new main path. Replacing x, α, y, β, δ by x_0, α_0, y_0, β_0, δ_0 in the vectors ${}^{3}_{x}V_i$, ${}^{2}_{y}V_i$ in eq.(8-2) we directly find the position of the new main path in the profile plane i+1 by the first three and two elements of the vectors ${}^{3}_{x}V_{i+1}$, ${}^{2}_{y}V_{i+1}$ respectively.

The next question to be answered is to find the position of a particle trajectory in the profile plane i+1 relative to the new main path. In other words we want to find Δx_{i+1}, $\Delta\alpha_{i+1}$, Δy_{i+1}, $\Delta\beta_{i+1}$ from Δx_i, $\Delta\alpha_i$, Δy_i, $\Delta\beta_i$, $\Delta\delta_i$.

This is possible if we change our transfer matrix ${}^3_xT$ and ${}^2_yT$ appropriately so that they contain the information about the new main path. As pointed out above we can not expect that such an approximation is correct to third order as in the original case describing a particle trajectory relative to the circular main path. Thus we should settle for a second order transfer matrix in the plane of deflection and for a first order transfer matrix in the transverse direction. These transfer matrices are found by algebraically introducing eqs.(8-3) into vectors ${}^3_xV_i$ and ${}^2_yV_i$ of eqs.(8-2), applying them to ${}^3_xT_i$ and ${}^2_yT_i$, rearranging the resultant vectors ${}^3_xV_{i+1}$ and ${}^2_yV_{i+1}$, according to the relation,

$$ {}^2_x\Delta V_{i+1} = {}^2_x\bar{T}\cdot{}^2_x\Delta V_i \quad , \tag{8-4a} $$

$$ {}^1_y\Delta V_{i+1} = {}^1_y\bar{T}\cdot{}^1_y\Delta V_i \quad . \tag{8-4b} $$

where

$$ {}^2_x\Delta V_i = (\Delta x,\ \Delta\alpha,\ \Delta\delta,\ \Delta x^2,\ \Delta x\Delta\alpha,\ \Delta x\Delta\delta,\ \Delta\alpha^2,\ \Delta\alpha\Delta\delta,\ \Delta\delta^2,\ \Delta y^2,\ \Delta y\Delta\beta,\ \Delta\beta^2)_j \tag{8-5a} $$

$$ {}^1_y\Delta_i = (\Delta y,\ \Delta\beta)_j \tag{8-5b} $$

The elements $\bar{R}(m|n)$ of the transfer matrices ${}^2_x\bar{T}$ and ${}^1_y\bar{T}$ are functions of the values of the position of the new main path x_0, α_0, y_0, β_0, δ_0 at the i-th profile plane and of the elements $R(m|n)$ of the transfer matrices ${}^3_xT_i$, and ${}^2_yT_i$ given in eqs.(8-2). In detail these elements are found to be :

$$ \bar{R}(x_i|x_j) = R(x_i|x_j) + \sum_{k=1}^{3} R(x_i|x_jx_k)(1+\delta_{jk})x_{0k} + \sum_{k=1}^{3}\sum_{\ell=1}^{k} R(x_i|x_jx_kx_\ell)(1+\delta_{jk}+\delta_{j\ell})x_{0k}x_{0\ell} $$

$$ \bar{R}(x_i|x_jx_k) = R(x_i|x_jx_k) + \sum_{\ell=1}^{3} R(x_i|x_jx_kx_\ell)(1+\delta_{jk}+\delta_{k\ell})x_{0\ell} \tag{8-6} $$

where x_1, x_2, x_3 stand for x, α and δ or τ, respectively, and δ_{jk}, $\delta_{k\ell}$, $\delta_{j\ell}$ are Kronecker's δ symbols. Up to this point we have listed only the first two lines of the matrix ${}^2_x\bar{T}$. The third line of this matrix contains only zeros except for $\bar{R}(\Delta\delta|\Delta\delta)=1$. The elements of the fourth to the twelfth line are derived according to eqs.(5-43). If the system consists of several fields we must first obtain the numerical values of each vector of the main path $(x_0,\ \alpha_0,\ \delta_0,\ y_0,\ \beta_0)_i$ relative to the old circular main path in a third order approximation. Introducing these values of each position vector and the corresponding elements of normal transfer matrix into the relations of eq.(8-6), we can derive each new transfer matrix relative to the new main path. If the system contains both electric and magnetic fields, the necessary "element change matrix" discussed in section 7.2. should be introduced in the suitable position.

8.2. Application of the Transfer Matrices ${}^{2}_{x}\bar{T}$ and ${}^{1}_{y}\bar{T}$

8.2.1. The Design of an Instrument with Reduced Image Aberrations

Using new transfer matrices ${}^{2}_{x}\bar{T}$, ${}^{1}_{y}\bar{T}$ it may be easier to construct a particle spectrometer that has reduced image aberrations since one has more parameters x_0, α_0, δ_0, y_0, β_0, even if only a limited range of variation is allowed. This application should be of high importance since in this manner a normal sector field spectrometer can be designed to have a higher performance though only small changes in the system are necessary. Here it is assumed that neither the geometry nor the field strength of any of the sector field is changed.

8.2.2. The First and Second Order Adjustment of a Geometrically Fixed Instrument

Let us assume that a particle spectrometer has been already constructed. A particle moving along the circular main paths in any of the sector fields is characterized by $x_0=\alpha_0=y_0=\beta_0=\delta_0=0$. In order to improve its properties we would like to change the position x_0, y_0 as well as the angles of inclination α_0, β_0 and the field strength δ_0. We can choose the value of δ_0 by two method, that is, by changing the accelerating potential or by changing the field strength. In the former case δ_0 is defined by the relation of eq.(5-6) as $\delta_0=(U-U_0)/U_0$. In the latter case we define the new quantity d which expresses the relative field strength as $d=(E-E_0)/E_0$. Then we obtain $\delta_0=\frac{-d}{1+d}$ for the particle having energy U_0. When we change both accelerating voltage and field strength we find

$$\delta_0 = \frac{\delta'-d}{1+d} \qquad (8\text{-}7)$$

where

$$\delta' = \frac{U-U_0}{U_0}$$

It should be noticed that we need not vary the elements of transfer matrix at all. When we continue the calculation to the other field, the value of d in eq.(8-7) should be modified.

The process of the calculation is as follows. First, according to the method stated in section (8.2.) we obtain the position vector of main beam relative to old circular main path and the new transfer matrix of the field considered, [see eq.(8-5)]. This calculation should be done for each field successively. Multiplying the new matrices of each field in turn, the overall transfer matrix can be determined numerically as a function of initial position vector, accelerating potential and field strength.

Selecting the suitable values of above parameters it becomes possible to control the position of main beam. Since we have many parameters we may be able to find out much better focusing conditions. An example will be shown in the next section.

8.3. Practical Application to r^{-1} High Resolution Mass Spectrometer

A high resolution mass spectrometer was constructed in our laboratory which has an r^{-1} magnetic field as the dispersing field together with a toroidal electric field and a uniform magnetic field as the focusing fields.[8)] The highest value of resolution obtained was one million by using a photo-plate. In fig.7 the block diagram of the system is shown. When designing the apparatus, the treatment of fringing field was not perfect, that is, the double focusing condition is not satisfied strictly under the present geometrical parameters. The present image aberration coefficients are listed in table.(2).

The behavior of the aberration coefficients have been investigated varying the strength of dispersing field and that of uniform field within realizable limits under the condition that the electric field is kept constant. The position of ion source and detector have been placed at the originally designed points. The relative field strength of the first magnetic field (r^{-1} field) is defined as d_1 and that of the second field d_2. Thus $d_1=d_2=0$ means the original circular main path. The results of calculation for the first order coefficients are shown in fig.8-a. The magnitude of the coefficients is the ordinate and d_2 the abscissa. The group of positive gradient lines correspond to direction focus and those of negative ones to energy focus. d_1 works as a parameter. To find suitable combinations of d_1 and d_2 the final image position is also shown in fig.8-b as functions of d_1 and d_2. The ordinate of fig.7-b equals to the distance of the main path from the geometrical center at the detector. Since the position of the detector can not be moved, we must impose such relation between d_1 and d_2 that the main path crosses the geometrical center at the detector even if both pathes differ to each other in the intermediate stages. From fig.7-b we can find that such conditions are satisfied at the neighbourhood of $d_1=-0.001$ and $d_2=0.008$ or $d_1=-0.002$ and $d_2=0.015$.

If we look at the same region of fig.8-a it is clear that the first order direction and energy focusing coefficients are nearly equal to zero. The change of c-value of the toroidal electric field causes also the change of the first order coefficients and this effect is also shown in fig.8-a. By systematical computer researches the best conditions for the double focusing are determined to be $d_1=-0.0015$, $d_2=0.010$, and $c=1.925$. Under this condition the position of main beam at the exit of the dispersing field is calculated to be 3mm. At this position of the mass spectrograph there is a monitor electrodes which can be inserted to the beam region and the position of the beam center at best focus was measured experimentally. The calculated beam position agrees with the experimentally measured value.

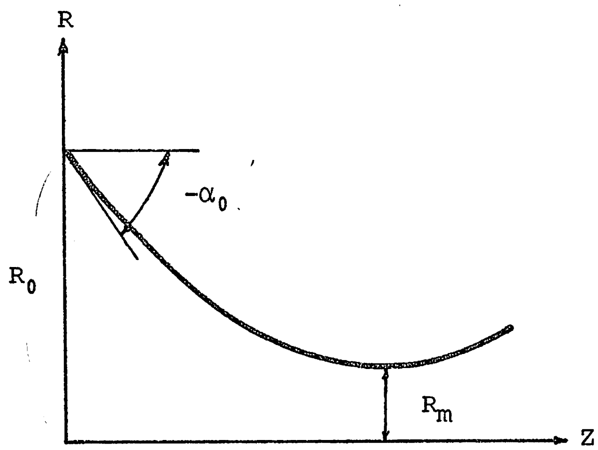
The variations of the second order aberration coefficients are also shown in column III of fig. 8. Relatively large change of the magnitude are seen when the position of main path is shifted. Second order coefficient $R_{\alpha\alpha}$, $R_{\alpha\delta}$, $R_{\delta\delta}$ of r^{-1} mass spectrometer were measured experimentally[11] and these values are listed in column IV. Considerably good agreement are seen between them. It may be given as a conclusion that though the adjustment of the aberration coefficients is possible by taking noncircular main path, it would be better to design the system so that the magnitude of the image aberrations stays always small even if the main beam passes along non-circular main path.

Space Charge and Relativistic Effect

In the whole discussions we have neglected the influence of the interaction between charged particles. It is worthwhile to examine the consequences of electromagnetic interaction between particles on beam focus, and to give the limit of the beam current. Consider an almost parallel beam of revolution symmetry, of radius R, and carrying current I with acceleration voltage V. The particles in the beam carry charge e and are of mass m. The electrical Coulomb force tends to make the beam divergent, whereas the magnetic force tends to make it converge, since two parallel currents in the same direction attract each other. We shall content ourselves with nonrelativistic calculations, which incidentally is very frequently enough. Since the force of magnetic origin is negligible, the radial motion of eqation is then

$$m\frac{d^2R}{dt^2} = \frac{em^{\frac{1}{2}}}{2\pi\varepsilon_0(2eV)^{\frac{1}{2}}}\frac{1}{R} \qquad (8\text{-}8)$$

We solve this equation under the initial condition that the distance to the axis is R_0 and the slope is $-\alpha_0$ at abscissa z=0. Then, the slope will diminish and disappear at the abscissa, where the beam will have a minimum radius R_m and dR/dt=0. By integration, eq. (8-8) give us



$$\frac{1}{2}m\left[\left(\frac{dR}{dt}\right)^2 - \left(\frac{dR}{dt}\right)_0^2\right] = \frac{(me)^{\frac{1}{2}}}{2(2)^{\frac{1}{2}}\pi\varepsilon_0}\frac{I}{V^{\frac{1}{2}}}\log\left(\frac{R}{R_0}\right) \qquad (8\text{-}9)$$

The Gauss theorem provides the radial electric field exerted on the edge of the beam

$$E = \frac{I}{2\pi\varepsilon_0 v}\frac{1}{R} \quad , \qquad (8\text{-}10)$$

where v is the velocity of particle and equals to $(2eV/m)^{\frac{1}{2}}$. Then the electric force F_e equals to

$$F_e = eE$$

The magnetic field can be easily derived by the Ampere theorem as:

$$B = \frac{\mu_0 I}{2\pi R}$$

The magnetic conversing force equals to

$$F_m = evB$$

The ratio of the magnetic force and the electric force is

$$\frac{F_m}{F_e} = \frac{v^2}{c^2} << 1$$

The minimum radius of the beam is thus

$$R_m = R_0 \exp\left[-2\pi\, \varepsilon_0 \left(\frac{2e}{m}\right)^{\frac{1}{2}} \frac{V^{\frac{3}{2}}}{I}\right]\alpha_0^2 \qquad (8\text{-}11)$$

In Table 3 the ratio R_m/R_0 calculated from the above equation for various conditions are given. It is evident that the influence of space charge effect is negligibly small in the case of the current intensity less than 10^{-8} Ampere.

We shall now finally give a simple consideration of the way in which the image forming property of the field is affected when the speed becomes faster so that relativistic treatment is needed. In the case of a mangetic field only, there is no change in the field constant k and accordingly no change in the focusing conditions. However, there is a change in the coefficient of dispersion with respect to momentum. Then we shall confine ourselves to the electric field only. In the relativistic case the field constant k should be modified to k*,

$$k^* = \left[k^2 - \left(\frac{v_0}{c}\right)^2\right]^{\frac{1}{2}} \qquad (8\text{-}12)$$

It follows naturally from this equation that the angle of deflection $\Phi = \pi/k^*$, where focusing takes place, varies according to the velocity v_0 of the ion following the central path. We calculate the relative change of deflection angle.

$$\varepsilon = \frac{\Phi^* - \Phi}{\Phi} = [\frac{k^2}{\sqrt{k^2-(\frac{v}{c})^2}} - 1] = \frac{eV}{k^2mc^2} \qquad (8\text{-}13)$$

From this equation, if we assume that the mass number is 100 and acceleration voltage is 100kV, then ε is estimated to be less than 10^{-6}. Then it leads to the conclusion that we can neglect the relativistic influence in the case of the energy less than 100keV.

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List of Figures and Tables

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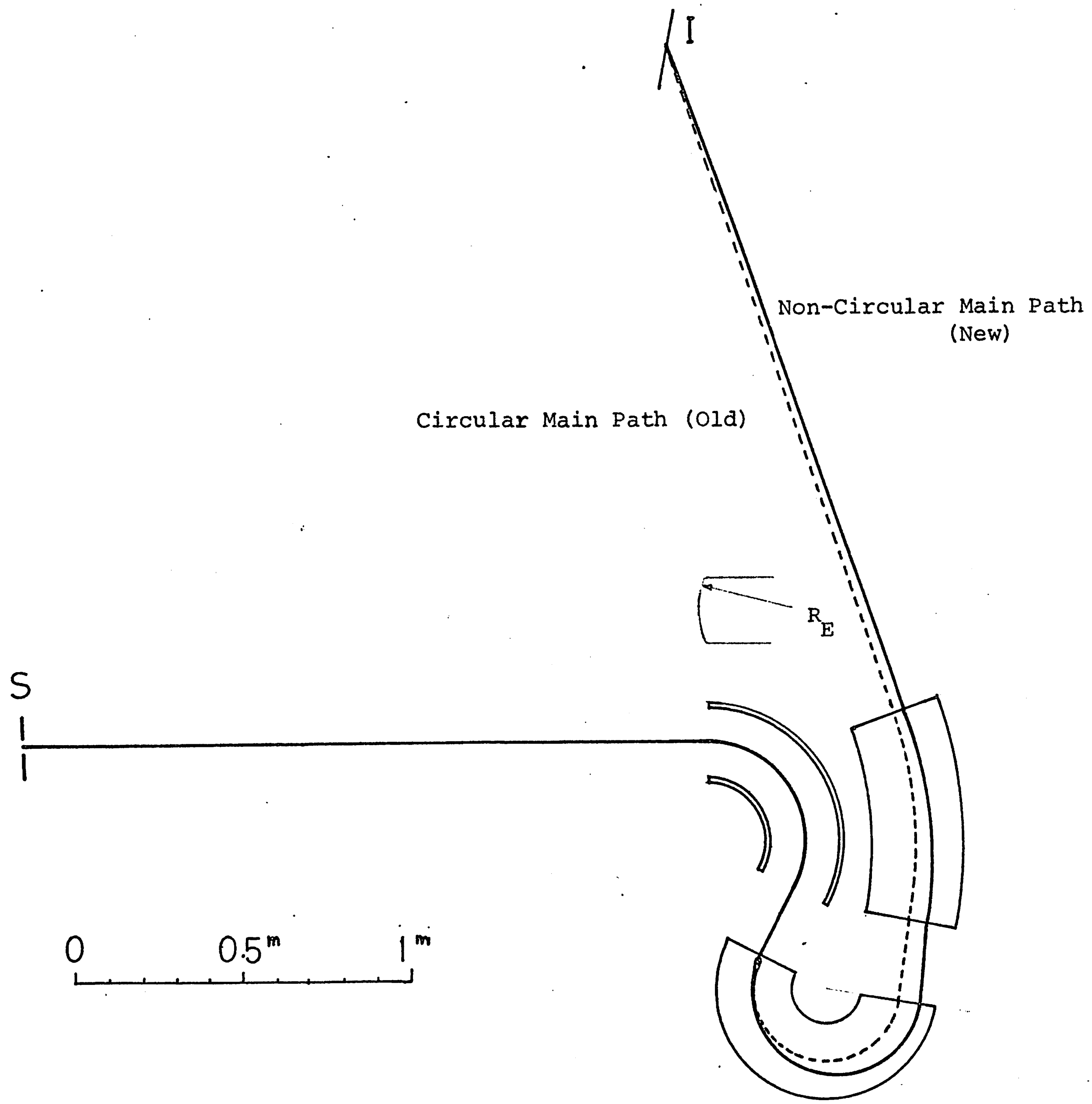


Fig.7. Schematic diagram of the ion path along non-circular main path.

Fig.8-a

The first order coefficients of r^{-1} mass spectrometer

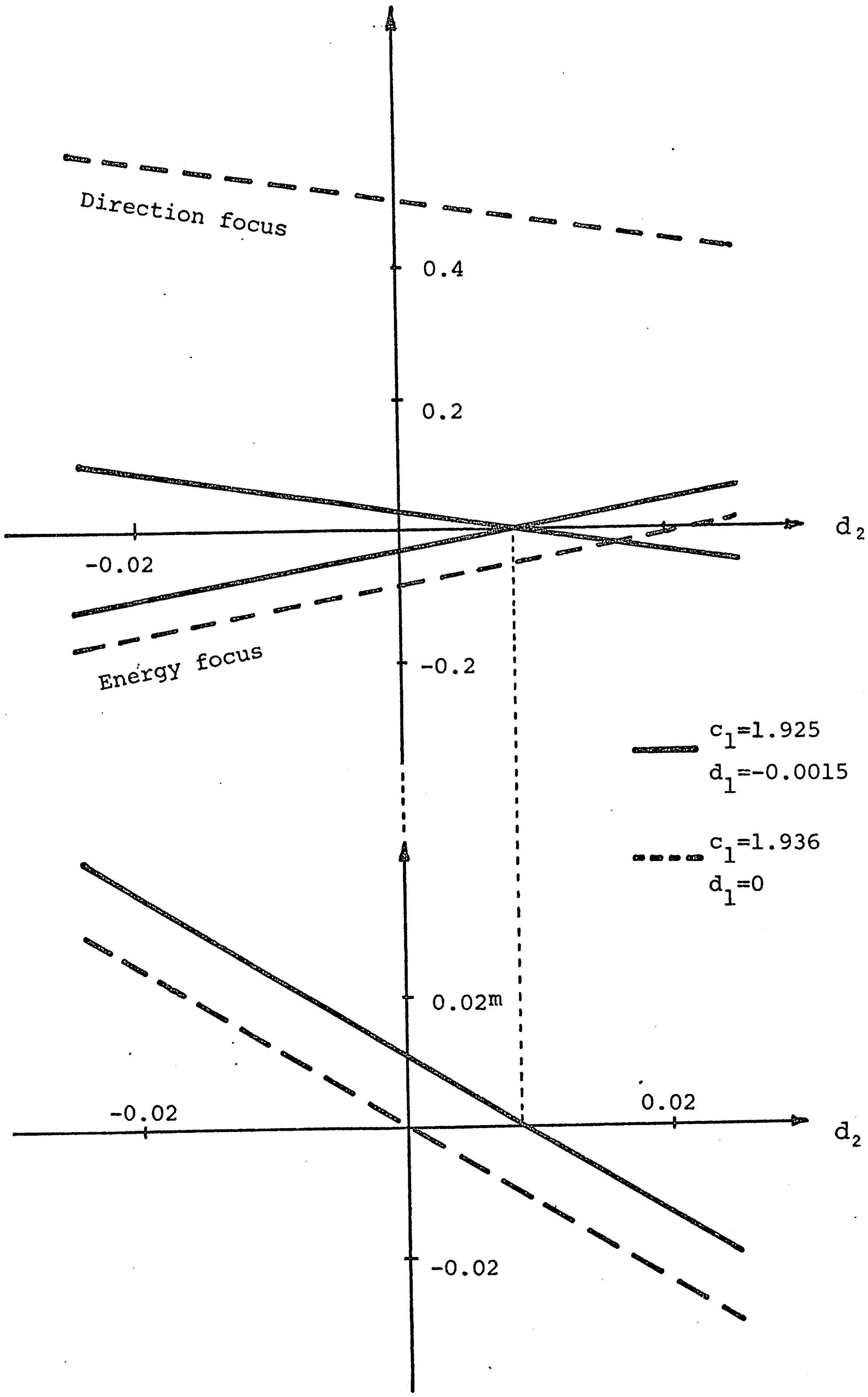
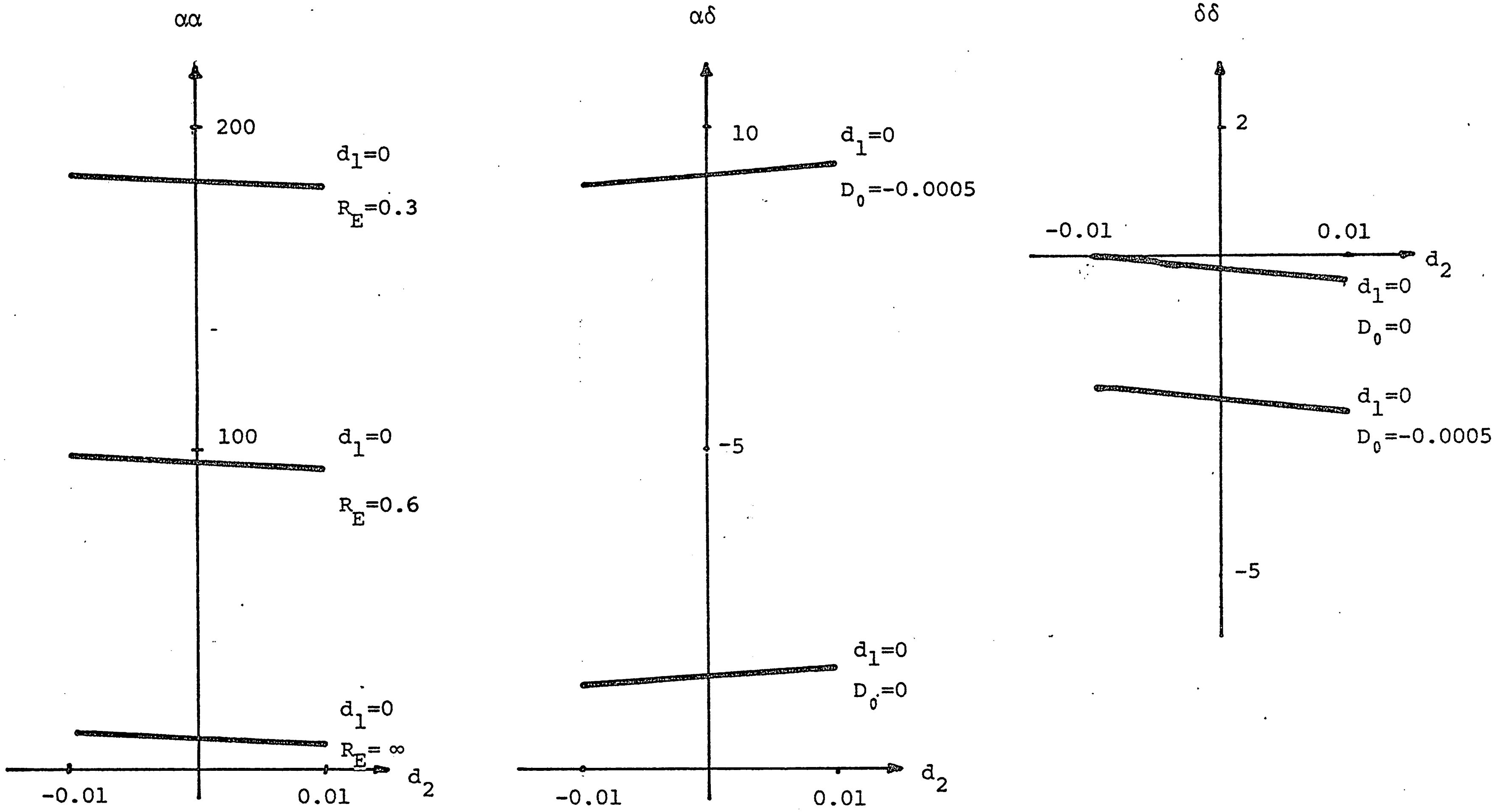


Fig.8-b

The final image posision

Fig.9 The second order coefficient of r^{-1} mass spectrometer



R_E: Axial Curvature of Electrode (See Fig. 7) D_0:Relative Change of Acceleration Energy

Table 1

Abbreviations used throughout this article

Symbol	Meaning
k_x	$\sqrt{2-c}$
k_y	$\sqrt{c}$
q_1	$-(6 + 4a_{20} + \frac{1}{2}a_{30})$
q_2	$-(4 + 6a_{20} + 2a_{30} + \frac{1}{6}a_{40})$
q_3	$-(1 - a_{20} - \frac{1}{2}a_{30} - \frac{1}{2}a_{40})$
q_4	$-\frac{1}{2}(1 - a_{20} - a_{30})$
q_5	$k_x^2 - k_y^2$
q_6	$k_x^2 - 2k_y^2$
q_7	$k_x^2 - 4k_y^2$
q_8	$-(3k_x^2 + 2q_1)$
q_9	$-(k_x^2 + 1 + 2q_1/k_x^2)$

Table 2. Image Aberration Coefficients

	R_x	R_α	R_δ	$R_{\alpha\alpha}$	$R_{\alpha\delta}$	$R_{\delta\delta}$	R_{yy}	$R_{y\beta}$	$R_{\beta\beta}$
I	-1.01	0.00	0.00	0.0	0.0	0.0	-0.16	-1.0	0.0
II	-0.802	0.50	-0.15	183.0	2.95	-0.90	-42.0	-178.0	-190.0
III	-0.99	0	0	60	12	-3	-12	-50	-50
IV	—	0	0	58	22	-18	—	—	—

I neglecting fringing field effect

II Considering fringing field effect

III choosing suitable main path

IV experimentally measured values

Table 3. Space Charge effect

ACCELERATION VOLTAGE (KV)	MASS (U)	ANGLE (RAD)	CURRENT (A)	R_m/R_0
10	10	0.01	10^{-6}	0.783
10	10	0.01	10^{-8}	0.0
10	10	0.01	10^{-10}	0.0
10	10	0.01	10^{-12}	0.0
10	10	0.001	10^{-6}	0.997
10	10	0.001	10^{-8}	0.783
10	10	0.001	10^{-10}	0.0
10	10	0.001	10^{-12}	0.0
10	100	0.01	10^{-6}	0.925
10	100	0.01	10^{-8}	0.0
10	100	0.01	10^{-10}	0.0
10	100	0.01	10^{-12}	0.0
10	100	0.001	10^{-6}	0.999
10	100	0.001	10^{-8}	0.925
10	100	0.001	10^{-10}	0.0
10	100	0.001	10^{-12}	0.0
100	10	0.01	10^{-6}	0.0
100	10	0.01	10^{-8}	0.0
100	10	0.01	10^{-10}	0.0
100	10	0.01	10^{-12}	0.0
100	10	0.001	10^{-6}	0.925
100	10	0.001	10^{-8}	0.0
100	10	0.001	10^{-10}	0.0
100	10	0.001	10^{-12}	0.0
100	100	0.01	10^{-6}	0.0
100	100	0.01	10^{-8}	0.0
100	100	0.01	10^{-10}	0.0
100	100	0.01	10^{-12}	0.0
100	100	0.001	10^{-6}	0.975
100	100	0.001	10^{-8}	0.0
100	100	0.001	10^{-10}	0.0
100	100	0.001	10^{-12}	0.0